MA 109 D1&D2 Lecture 13

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The formula for arc length

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where "$\sim$" means approximately equal. We can use this idea to define the arc length as

$$S := \lim_{\Delta x_i \to 0} \sum_{i=1}^{\infty} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

provided this limit exists (in particular, we demand that the limit is a finite number).

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Example: Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the curve given by $\gamma(t) = (t, f(t))$, where

$$f(t) = \begin{cases} t\cos\left(\frac{\pi}{2t}\right), \text{ if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$$

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is correct, you should be able to check that this curve has infinite arc length. Try it as an exercise.

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is correct, you should be able to check that this curve has infinite arc length. Try it as an exercise.

Notice that the curve above is given by a continuous function. Curves for which the arc length S is finite are called rectifiable curves. You can easily check that the graphs of piecewise $\mathcal{C}^1$ functions are rectifiable.

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In fact, there exist space filling curves, that is curves $\gamma : [0, 1] \to [0, 1] \times [0, 1]$ which are continuous and surjective. Obviously the graph of this curve "fills up" the entire square. Such curves are not rectifiable (can you prove this?)

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The existence of such curves should make you question whether your intuitive notion of dimension actually has any mathematical basis. If a line segment can be mapped continuously onto a square, is it reasonable to say that they have different dimensions? After all, this means we can describe any point on the square using just one number.

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We will answer this question (without a proof) in MA 111 (maybe). We will also come back to the arc length of a curve when studying multivariable calculus.

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for some constant C. Set $x = 1$ to obtain $C = -f(y)$. Thus,

$$f(xy) = f(x) + f(y).$$

The logarithm and exponential functions

The function $f(x)$ is usually denoted $\ln x$. Since $f'(x) = \frac{1}{x} > 0$, whenever $x > 0$, we see that f is (strictly) monotonic increasing and concave.

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It is not hard to see that f must have an inverse function. This is the exponential function sometimes denoted $\exp(x)$. Clearly $\exp(x + y) = \exp(x) \cdot \exp(y)$. Again, it requires some work to see that

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When $x = 1$ we will obtain a formula for e!

The Mean Value Theorem for Integration

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This says that there exists $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c).$$

But this is the same as saying

$$\int_a^b f(t)dt = f(c)(b - a).$$

This is the Mean Value Theorem for integration.

Functions with range contained in $\mathbb{R}$

We will be interested in studying functions $f : \mathbb{R}^m \to \mathbb{R}$, when $m = 2, 3$. We have already mentioned how limits of such functions can be studied in the first few lectures. Before doing this in detail, however, we will study certain other features of functions in two and three variables.

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The most basic thing one needs to understand about a function is the domain on which it is defined. Very often a function is given by a formula which makes sense only on some subset of $\mathbb{R}^m$ and not on the whole of $\mathbb{R}^m$. When studying functions of two or more variables given by formulæ it makes sense to first identify this subset, which is sometimes call the natural domain of the function, and to describe it geometrically if possible.

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The natural domain is thus

$$\mathbb{R}^2 \setminus \{(x, y) \,|\, x^2 - y^2 = 0\},$$

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This function is defined whenever $x^2 + y^2 \neq 0$, in other words, in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Level curves and contour lines

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One way is to study the level sets of the functions. These are the sets of the form $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$, where c is a constant. The level set "lives" in the xy-plane.

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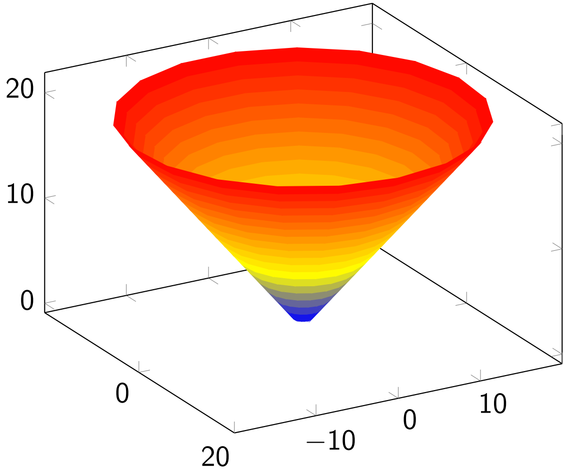
I have a couple of pictures in the next two slides to illustrate the point.



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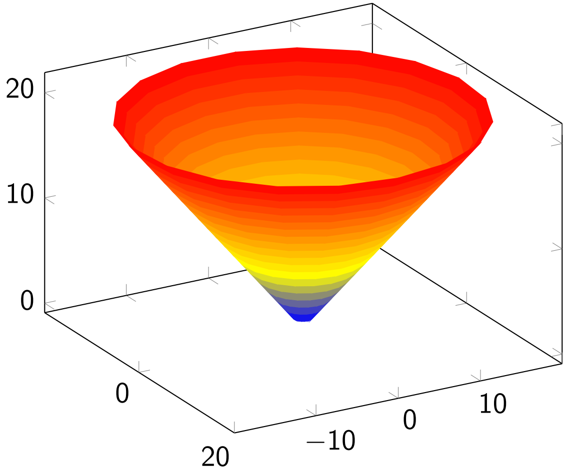
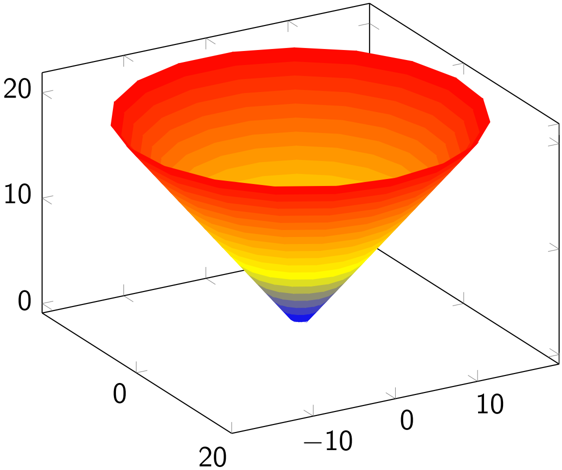


This is the graph of the function $z = \sqrt{x^2 + y^2}$ lying above the xy-plane. It is a right circular cone.



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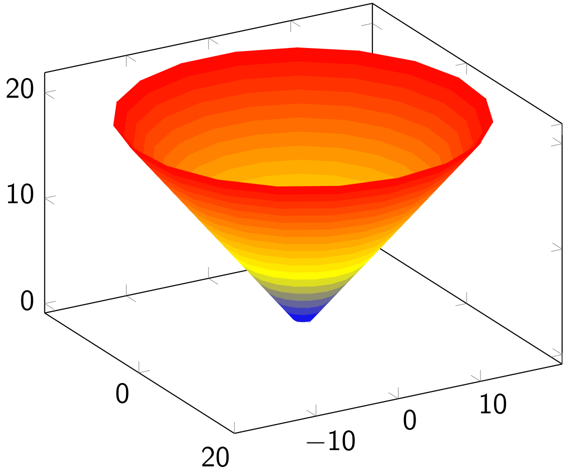
The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.



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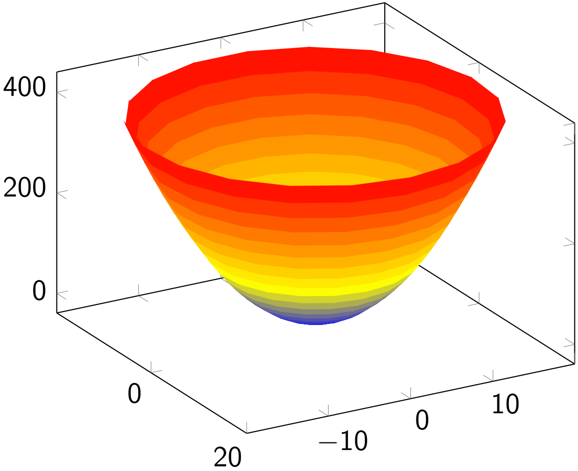
The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.The curves given by $z = f(x, 0)$ and $z = f(0, y)$ give pairs of straight lines in the planes $y = 0$ and $x = 0$.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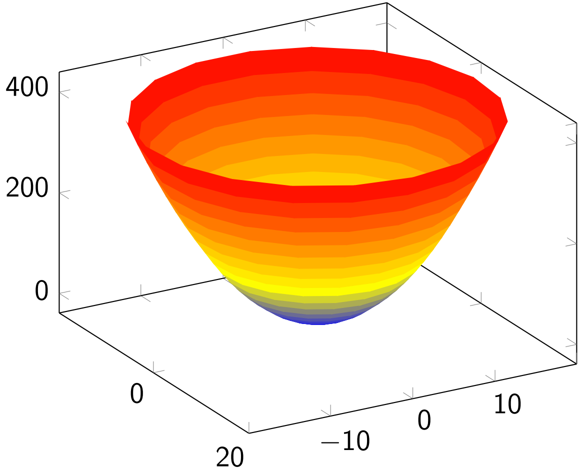

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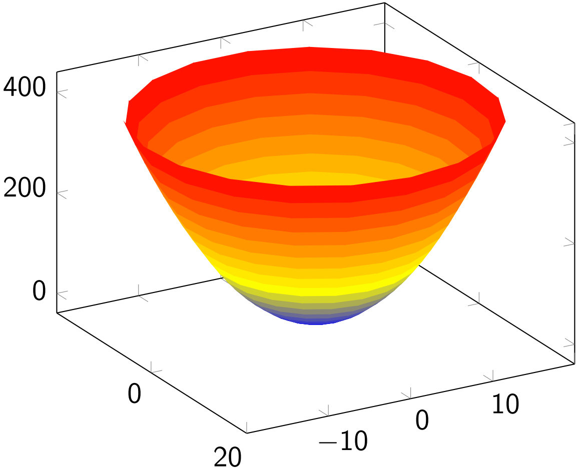

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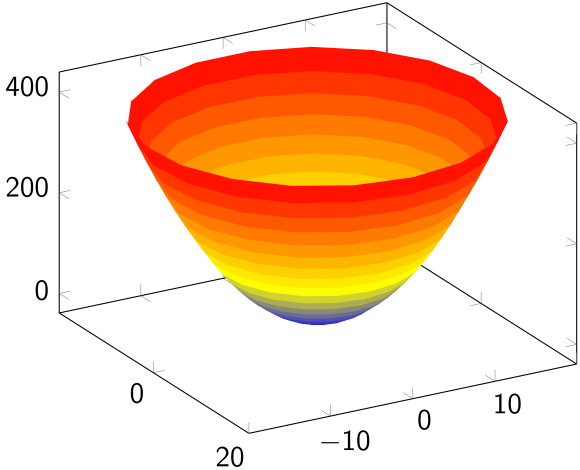

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The contour lines $z = c$ give circles lying on planes parallel to the xy-plane.The curves $z = f(x, 0)$ or $z = f(y, 0)$ give parabolæ lying in the planes $y = 0$ and $x = 0$. Exercise 5.2.(ii).

Limits

We have already said what it means for a function of two or more variables to approach a limit. We simply have to replace the absolute value function on $\mathbb{R}$ by the distance function on $\mathbb{R}^m$. We will do this in two variables. The three variable definition is entirely analogous. We will denote by U a set in $\mathbb{R}^2$.

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Definition: A function $f : U \to \mathbb{R}$ is said to tend to a limit l as $x = (x_1, x_2)$ approaches $c = (c_1, c_2)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon,$$

whenever $0 < \|x - c\| < \delta$.

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We recall that

$$\|x\| = \sqrt{x_1^2 + x_2^2}.$$

Continuity

Before talking about continuity we remark the following. In the plane $\mathbb{R}^2$ it is possible to approach the point c from infinitely many different directions - not just from the right and from the left. In fact, one may not even be approaching the point c along a straight line! Hence, to say that a function from $\mathbb{R}^2$ to $\mathbb{R}$ possesses a limit is actually imposing a strong condition - for instance, the limits along all possible curves leading to the point must exist and all these (infinitely many) limits must be equal.

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Definition: The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

The rules for limits and continuity

The rules for addition, subtraction, multiplication and division of limits remain valid for functions of two variables (or three variables for that matter). Nothing really changes in the statements or the proofs.

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Using these rules, we can conclude, as before, that the sum, difference, product and quotient of continuous functions are continuous (as usual we must assume that the denominator of the quotient is non zero).

Continuity through examples

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Exercise 5.3.(i) asks whether the function

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^6+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

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is continuous at $(0,0)$.

Solution: Let us look at the sequence of points $z_n = (\frac{1}{n}, \frac{1}{n^3})$, which goes to 0 as $n \to \infty$. Clearly $f(z_n) = \frac{1}{2}$ for all n, so

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This shows that f is not continuous at 0.
But does the limit exist?

Iterated limits

When evaluating a limit of the form $\lim_{(x_1,x_2)\to(c_1,c_2)} f(x_1,x_2)$ one may naturally be tempted to let x_1 go to c_1 first, and then let x_2 go to c_2. Does this give the limit in the previous sense?

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Exercise 5.5: Let

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we have

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0} 0 = 0$$

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Similarly, one has $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$.

However, choosing $z_n = (\frac{1}{n},\frac{1}{n})$, shows that $f(z_n) = 1$ for all $n \in \mathbb{N}$. Now choose $z_n = (\frac{1}{n},\frac{1}{2n})$ to see that the limit cannot exist.