MA 105 D1 &D2 Lecture 14

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Recap: Limits of functions of severable variables

Limits and continuity

Differentiation

Ideas introduced yesterday

1. Natural domain

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2. Level sets

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3. Contour lines

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5. Continuity

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Definition: The function $f : U \to \mathbb{R}$ is said to be continuous at a point c if

$$\lim_{x \to c} f(x) = f(c).$$

The rules for limits and continuity

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Using these rules, we can conclude, as before, that the sum, difference, product and quotient of continuous functions are continuous (as usual we must assume that the denominator of the quotient is non zero).

Partial Derivatives

As before, U will denote a subset of $\mathbb{R}^2$. Given a function $f : U \to \mathbb{R}$, we can fix one of the variables and view the function f as a function of the other variable alone. We can then take the derivative of this one variable function.

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Definition: The partial derivative of $f : U \to \mathbb{R}$ with respect to x_1 at the point (a, b) is defined by

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Similarly, one can define the partial derivative with respect to x_2. In this case the variable x_1 is fixed and f is regarded only as a function x_2:

$$\frac{\partial f}{\partial x_2}(a, b) := \lim_{x_2 \to b} \frac{f((a, x_2)) - f((a, b))}{x_2 - b}.$$

Directional Derivatives

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Definition: The directional derivative of f in the direction v at a point $x = (x_1, x_2)$ is denoted by $\nabla_v f(x)$ and is defined as

$$\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = \lim_{t \to 0} \frac{f((x_1 + tv_1, x_2 + tv_2)) - f((x_1, x_2))}{t}.$$

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If we take $v = (1, 0)$ in the above definition, we obtain $\frac{\partial f}{\partial x_1}(x)$, while $v = (0, 1)$ yields $\frac{\partial f}{\partial x_2}(x)$.

Consider the function

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It should be clear to you that since this function is constant along the two axes,

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On the other hand, $f(x_1, x_2)$ is not continuous at the origin! Thus, a function may have both partial derivatives (and, in fact, any directional derivative - see the next slide) but still not be continuous. This suggests that for a function of two variables, just requiring that both partial derivatives exist is not a good or useful definition of "differentiability".

Recall again, the following function from Exercise 5.5:

$$\frac{x^2y^2}{x^2y^2 + (x-y)^2} \quad \text{for} \quad (x,y) \neq (0,0).$$

Let us further set $f(0,0) = 0$. You can check that every directional derivative exists and is equal to 0, except along $y = x$ when the directional derivative is not defined. However, we have already seen that the function is not continuous at the origin since we have shown that $\lim_{(x,y)\to 0} f(x,y)$ does not exist. For an example with directional derivatives in all directions see Exercise 5.3(i).

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Conclusion: All directional derivatives may exist at a point even if the function is discontinuous.

Let us go back and examine the notion of differentiability for a function of $f(x)$ of one variable. Suppose f is differentiable at the point x_0, What is the equation of the tangent line through $(x_0, f(x_0))$?

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If we consider the difference $f(x) - f(x_0) - f'(x_0)(x - x_0)$ we get the distance of a point on the tangent line from the curve $y = f(x)$. Writing $h = (x - x_0)$, we see that the difference can be rewritten

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The tangent line is close to the function f - how close?- so close that even after dividing by h the distance goes to 0. A few lectures ago we wrote this as

$$|f(x_0 + h) - f(x_0) - f'(x_0)h| = \varepsilon_1(h)|h|$$

where $\varepsilon_1(h)$ is a function that goes to 0 as h goes to 0.

The preceding idea generalises to two (or more) dimensions. Let $f(x, y)$ be a function which has both partial derivatives. In the two variable case we need to look at the distance between the surface $z = f(x, y)$ and its tangent plane.

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Let us determine the tangent plane to $z = f(x, y)$ passing through a point $P = (x_0, y_0, z_0)$ *on the surface*. In other words, we have to determine the constants a and b.

If we fix the y variable and treat $f(x, y)$ only as a function of x, we get a curve. Similarly, if we treat $g(x, y)$ as function only of x, we obtain a line. The tangent to the curve must be the same as the line passing through (x_0, y_0, z_0), and, in any event, their slopes must be the same. Thus, we must have

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial g}{\partial x}(x_0, y_0) = a.$$

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Arguing in exactly the same way, but fixing the x variable and varying the y variable we obtain

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Hence, the equation of the tangent plane to $z = f(x, y)$ at the point (x_0, y_0) is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Differentiability for functions of two variables

We now define differentiability for functions of two variables by imitating the one variable definition, but using the "$p(h)$" version.

We let $(x, y) = (x_0, y_0) + (h, k) = (x_0 + h, y_0 + k)$

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$$\lim_{(h,k)\to 0} \frac{\left| f(x_0 + h, y_0 + k) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k \right|}{\|(h, k)\|} = 0,$$

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This is saying that the distance between the tangent plane and the surface is going to zero even after dividing by $\|(h, k)\|$. We could rewrite this as

$$\left| f((x_0, y_0) + (h, k)) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)h - \frac{\partial f}{\partial y}(x_0, y_0)k \right| = p(h, k)\|(h, k)\|$$

where $p(h, k)$ is a function that goes to 0 as $\|(h, k)\| \to 0$.This form of differentiability now looks exactly like the one variable version case.

The derivative as a linear map

We can rewrite the differentiability criterion once more as follows. We define the 1×2 matrix

$$Df(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}.$$

A 1×2 matrix can be multiplied by a column vector (which is 2×1 matrix) to give a real number. In particular:

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$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \frac{\partial f}{\partial x}(x_0, y_0)h + \frac{\partial f}{\partial y}(x_0, y_0)k$$

that is,

$$Df(x_0, y_0) \begin{pmatrix} h \\ k \end{pmatrix} = \frac{\partial f}{\partial x}(x_0, y_0)h + \frac{\partial f}{\partial y}(x_0, y_0)k$$

The definition of differentiability can thus be reformulated using matrix notation.

Definition: The function $f(x,y)$ is said be differentiable at a point (x_0, y_0) if there exists a matrix denoted $Df((x_0, y_0))$ with the property that

$$f((x_0, y_0) + (h,k)) - f(x_0, y_0) - Df(x_0, y_0)\begin{pmatrix} h \\ k \end{pmatrix} = p(h,k)\|(h,k)\|,$$

for some function $p(h,k)$ which goes to zero as (h,k) goes to zero. Viewing the derivative as a matrix allows us to view it as a linear map from $\mathbb{R}^2 \to \mathbb{R}$. Given a 1×2 matrix A and two column vectors v and w, we see that

$$A \cdot (v + w) = A \cdot v + A \cdot w \quad \text{and} \quad A \cdot (\lambda v) = \lambda (A \cdot v),$$

for any real number λ. As we have seen before, functions satisfying the above two properties are called linear functions or linear maps. Thus, the map $v \to A \cdot v$ gives a linear map from $\mathbb{R}^2$ to $\mathbb{R}$.

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The matrix $Df(x_0, y_0)$ is called the Derivative matrix of the function $f(x, y)$ at the point (x_0, y_0).

The Gradient

When viewed as a row vector rather than as a matrix, the Derivative matrix is called the gradient and is denoted $\nabla f(x_0, y_0)$. Thus

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right).$$

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In terms of the coordinate vectors $\mathbf{i}$ and $\mathbf{j}$ the gradient can be written as

$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{i} + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{j}.$$

A criterion for differentiability

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Theorem 26: Let $f : U \to \mathbb{R}$. If the partial derivatives $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ exist and are continuous in a neighbourhood of a point (x_0, y_0) (that is in a region of the plane of the form $\{(x, y) \mid \|(x, y) - (x_0, y_0)\| < r\}$ for some $r > 0$. Then f is differentiable at (x_0, y_0).

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We omit the proof of this theorem. However, we note that a function whose partial derivatives exist and are continuous is said to be continuously differentiable or of class $\mathcal{C}^1$. The theorem says that every $\mathcal{C}^1$ function is differentiable.