MA 109: Quiz Review

Ravi Raghunathan

Department of Mathematics

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General Advice

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2. When trying to understand a definition, make sure you know plenty of examples.
3. When trying to understand a theorem, make sure you know counter-examples to the conclusion of the theorem when you drop some of the hypotheses.
4. In general, the statement of the theorem is more important than its proof. And examples are more important than theorems!

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The test is only about 1 hour long. Avoid using the bathroom during this time if possible.

Quiz instructions

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Read these instructions carefully before proceeding.

1. Answers unsupported by satisfactory reasoning may not be awarded marks. In "True or False" questions, you must give adequate justification if your answer is "True" and provide a counter-example if your answer is "False".
2. **Very important:** Let a denote the last digit of your roll number and let b be the second last digit of your roll number. Let $A = 10 - a$ and $B = 10 - B$. Note that $0 \leq a, b \leq 9$ and $1 \leq A, B \leq 10$. Record your values for a, A, b, B below. You must use these values of a, b, A, B in your quiz. If you use any other values, you will be immediately awarded 0 marks for that question.

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3. To prove that a sequence does not converge you have to show that no real number can be a limit. Thus you must take an arbitrary l and find some fixed $\epsilon > 0$ - this ϵ can be chosen to your convenience so that $|a_n - l| > \epsilon$ for infinitely many n.

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4. Theorems to remember for showing that limits exist: the sum, difference, product and quotient and the Sandwich Theorem. In this case you will already know that some sequence has a limit and deduce that another sequence has a limit by comparing it to the known one.

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Unless we explicitly mention that you must use the ϵ-N definition to prove that a limit exists, you do not have to. You may use the rules for limits and other theorems instead. You can use simple facts without proving them: e.g. $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$ if $\alpha > 0$.

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Since $\lim_{n\to\infty} a_n = 0$, for ϵ^2, the square of ϵ that we fixed earlier, there exists N_1 such that

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Then for $N = N_1$ we have

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The ideas behind proving or disproving the existence of limits are the same as for sequences (of course, there is no analogue of monotonic bounded sequences or Cauchy sequences).

You can use the basic limits you learnt in 11th/12th standard like $\lim_{x\to 0} \sin x/x = 1$.

Continuity

Of course, you have to know the definition. Again, unless asked do not use $\epsilon - \delta$. You may use basic facts about limits of functions to prove what you want.
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   The sum, difference, product etc. of continuous functions is continuous. The composition of continuous functions is continuous.

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2. Rolle's theorem and the MVT,

Know the basic examples and counter-examples: a function that is continuous but not differentiable, a function that is differentiable but not continuously differentiable.

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Since the given function is a cubic, $f(x) \to \pm\infty$ as $x \to \pm\infty$, hence again by IVP we get two more zeros of f in the intervals $(-\infty, -\sqrt{10/3})$ and $(\sqrt{10/3}, \infty)$.

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Alternate solution: Show that $f(x)$ changes sign three times. Note that $f(-10) < 0$, $f(-1) > 0$, $f(1) < 0$ and $f(10) > 0$.

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Further, the derivative, $4x^3 + 3$, is non-zero on $[-2, -1]$, so by Rolle's theorem, f has no more zeros in the given interval.

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Taylor's theorem: Know how to compute the Taylor polynomials. Know the form of the Remainder term. Recall that there are smooth functions for which the Taylor series about a point converges but does not converge to the function ($e^{-1/x}$).

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Remember the form of the remainder:

$$R_n(b) = \frac{f^{(n+1)}(c)(b-a)^{n+1}}{(n+1)!}$$

To estimate the remainder, you will need to bound $f^{(n+1)}(c)$ some number. Remember that $c \in (a, b)$. If $f^{(n+1)}(c)$ grows slower than $n!$, then by taking n large we can make the remainder small.

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The Fundamental Theorem of calculus.

Exercise 6

3. For the function $f(x) = 3x^2$ and the partition

$$P_n = \left\{0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1\right\}$$

of $[0, 1]$ find the lower sum, $L(f, P_n)$, upper sum, $U(f, P_n)$. Compute $\sup_n L(f, P_n)$ and $\inf_n U(f, P_n)$.

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Solution:

$$L(f, P_n) = \sum_{i=0}^{n-1} 3\frac{i^2}{n^2}\frac{1}{n} = 3\frac{1}{n^3}\frac{n(n-1)(2n-1}{6}$$

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$$\sup_n L(f, P_n) = 1 \quad \text{and} \quad \inf_n U(f, P_n) = 1.$$

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Thus, the given sum is the Riemann sum for the function $\dfrac{1}{x^2+1}$ over the interval $[0,1]$ with respect to the partition
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Since the function $1/(1+x^2)$ is continuous on $[0,1]$, it is Riemann integrable.

Hence the limit of the given sum is $\int_0^1 \frac{1}{x^2+1} dx = \pi/4.$