MA 109: D1&D2 Lecture 18

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Vector valued functions

Vector fields

Back to the derivative

Functions from $\mathbb{R}^m \to \mathbb{R}^n$

So far we have only studied functions whose range was a subset of $\mathbb{R}$. Let us now allow the range to be $\mathbb{R}^n$, $n = 1, 2, 3, \ldots$. Can we understand what continuity, differentiability etc. mean?

Let U be a subset of $\mathbb{R}^m$ $(m = 1, 2, 3, \ldots)$ and let $f : U \to \mathbb{R}^n$ be a function. If $x = (x_1, x_2, \ldots, x_m) \in U$, $f(x)$ will be an n-tuple where each coordinate is a function of x. Thus, we can write $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where each $f_i(x)$ is a function from U to $\mathbb{R}$.

Functions which take values in $\mathbb{R}$ are called scalar valued functions. Functions which take values in $\mathbb{R}^n$, $n > 1$ are usually called vector valued functions.

Continuity of vector valued functions

The definition of continuity is exactly the same as before.
Definition: The function f is said to be continuous at a point $c \in U$ if

$$\lim_{x \to c} f(x) = f(c).$$

How does one define the limit on the left hand side? The function

f takes values in $\mathbb{R}^n$, so its limit must be a point in $\mathbb{R}^n$, say $l = (l_1, l_2, \ldots, l_n)$.

Definition: We say that $f(x)$ tends to the limit l if given any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \|x - c\|_m < \delta$, then

$$\|f(x) - l\|_n < \epsilon.$$

You can easily prove the following theorem yourself:

Theorem: The function $f : U \to \mathbb{R}^n$ is continuous if and only if each of the functions $f_i : U \to \mathbb{R}$, $1 \leq i \leq n$, is continuous.

Vector fields

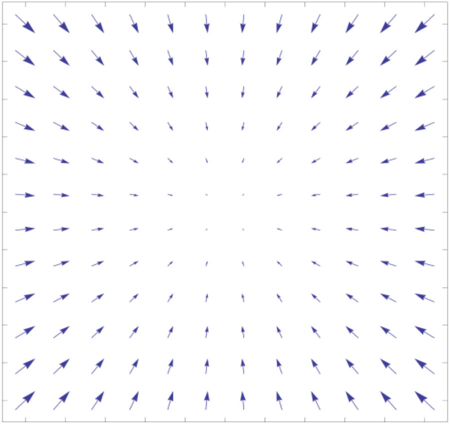
When $m = n$, vector valued functions are often called vector fields. We will study vector fields in slightly greater detail when $m = n = 2$ and $m = n = 3$.

We have already seen one example of a vector field - the gravitational force field $-\frac{GMm}{r^3} \cdot \mathbf{r}$ felt by a mass m whose position vector with respect to a mass M at the origin is $\mathbf{r}$. In this particular case we showed the the force field arose as the gradient of a scalar valued function (the potential $V = GMm/r$).

One of the most important questions in calculus is the following: Given a vector field, when does it arise as the gradient of a scalar function? In physics, vector force fields that arise from scalar potential functions are called conservative.

Some pictures of vector fields

We can actually visualize two dimensional vector fields as follows. At each point in $\mathbb{R}^2$ we can draw an arrow starting at that point pointing in the direction of the image vector and with size proportional to the magnitude of the image vector.



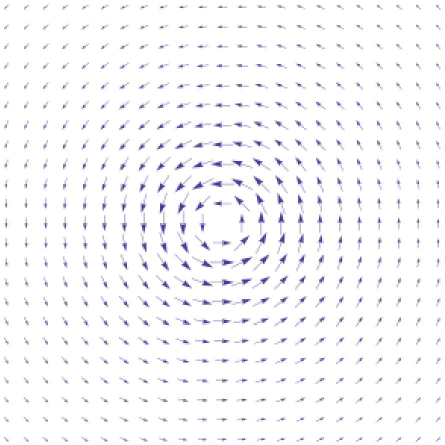
What function from $\mathbb{R}^2$ to $\mathbb{R}^2$ does this picture represent?

$$f(x, y) = (-x, -y)$$

the the radial vector field.

http://en.wikipedia.org/wiki/File:Radial_vector_field_dense.svg

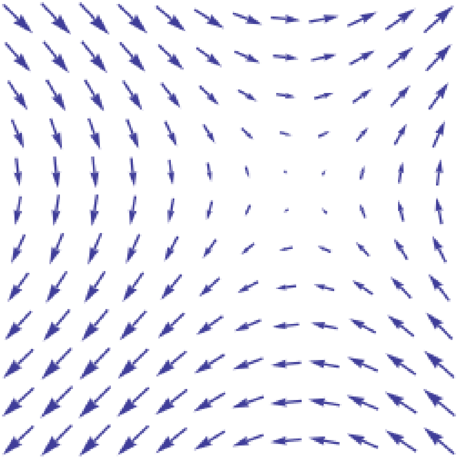
How about this one?



$$f(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

This is an example of an irrotational vector field.
It cannot be written as the gradient of a potential function.

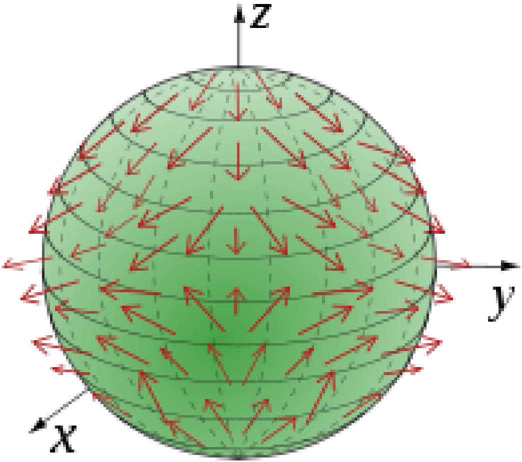
Here is another (more complicated one)



$$f(x, y) = (\sin y, \sin x)$$

http://en.wikipedia.org/wiki/File:VectorField.svg

One can also talk about ~~two dimensional~~ (struck out because it caused confusion in class) vector fields on any two dimensional surface. Here is a picture of a vector field on a sphere.



http://en.wikipedia.org/wiki/File:Vector_sphere.svg

Vector fields in the "real world"

Many real world phenomena can be understood using the language of vector fields. In physics, apart from gravitation, electromagnetic forces can also be represented by vector fields. That is, to each point in space we attach the vector representing the force at that point. Such fields are called force fields.

Fluids flowing are also often modeled using vector fields, with each point being mapped to the vector representing the velocity of the fluid flow. For instance, the velocity of winds in the atmosphere can be represented as a vector field. Such fields are called velocity fields.

A diversion: How to calculate powers of e in your head?

From Richard Feynman's "Surely you're joking Mr. Feynman!" (pages 173-174):

One day at Princeton I was sitting in the lounge and overheard some mathematicians talking about the series for e to the x power which is $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. Each term you get by multiplying the preceding term by x and dividing by the next number. For example, to get the next term after $x^3/3!$ you multiply that term by x and divide by 4. It's very simple.

When I was a kid I was excited by series, and had played with this thing. I had computed e using that series, and had seen how quickly the new terms became very small.

I mumbled something about how it was easy to calculate e to any power using that series (you just substitute the power for x).
"Oh yeah?" they said. "Well, the what's e to the 3.3?" said some joker - I think it was Tukey.
I say, "That's easy. It's 27.11."

Feynman's anecdote continued

Tukey knows it isn't so easy to compute all that in your head.
"Hey! How'd you do that?"
Another guy says, "You know Feynman, he's just faking it. It's not really right."
They go to get a table, and while they're doing that, I put on a few more figures: "27.1126," I say.
They find it in the table. "It's right! But how'd you do it!"
"I just summed the series."
"Nobody can sum the series that fast. You must just happen to know that one. How about e^3?"
"Look," I say. "It's hard work! Only one a day!"
"Hah! It's a fake!" they say, happily.
"All right," I say, "It's 20.085."
They look in the book as I put a few more figures on. They're all excited now, because I got another one right.

Here are these great mathematicians of the day, puzzled at how I can compute e to any power! One of them says, “He just can’t be substituting and summing - it’s too hard. There’s some trick. You couldn’t do just any old number like e to the 1.4.”

I say, “It’s hard work, but for you, OK. It’s 4.05.”

As they’re looking it up, I put on a few more digits and say, “And that’s the last one for the day!” and walk out.

What happened was this: I happened to know three numbers - the logarithm of 10 to the base *e* (needed to convert numbers from base 10 to base *e*), which is 2.3026 (so I knew that *e* to the 2.3 is very close to 10), and because of radioactivity (mean-life and half-life), I knew the log of 2 to the base *e*, which is .69315 (so I also knew that *e* to the .7 is nearly equal to 2). I also knew *e* (to the 1), which is 2.71828.

The first number they gave me was *e* to the 3.3, which is *e* to the 2.3 (10) times *e*, or 27.18. While they were sweating about how I was doing it, I was correcting for the extra .0026 - 2.3026 is a little high.

I knew I couldn't do another one; that was sheer luck. But then the guy said *e* to the 3: that's *e* to the 2.3 times *e* to the .7, or ten times two. So I knew it was 20.something, and while they were worrying how I did it, I adjusted for the .693.

Now I was sure I couldn't do another one, because the last one was again by sheer luck. But the guy said *e* to the 1.4 which is *e* to the .7 times itself. So all I had to do is fix up 4 a little bit!

They never did figure out how I did it.

https://en.wikipedia.org/wiki/Richard_Feynman Richard Feynman (1918-1988)

The derivative for $f : U \to \mathbb{R}^n$

We now define the derivative for a function $f : U \to \mathbb{R}^n$, where U is a subset of $\mathbb{R}^m$.

The function f is said to be differentiable at a point x if there exists an $n \times m$ matrix $Df(x)$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - Df(x) \cdot h\|}{\|h\|} = 0.$$

Here $x = (x_1, x_2, \ldots, x_m)$ and $h = (h_1, h_2, \ldots h_m)$ are vectors in $\mathbb{R}^m$.

The matrix $Df(x)$ is usually called the total derivative of f. It is also referred to as the Jacobian matrix. What are its entries?

From our experience in the 2×1 case we might guess (correctly!) that the entries will be the partial derivatives.

Here is the total derivative or the derivative matrix written out fully.

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_m}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & \cdots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix}$$

In the 2×2 case we get

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{pmatrix}.$$

As before, the derivative may be viewed as a linear map, this time from $\mathbb{R}^m$ to $\mathbb{R}^n$ (or, in the case just above, from $\mathbb{R}^2$ to $\mathbb{R}^2$).

Rules for the total derivative

Just like in the one variable case, it is easy to prove that

$$D(f+g)(x) = Df(x) + Dg(x).$$

Somewhat harder, but only because the notation gets more cumbersome, is the Chain rule:

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x),$$

where $\circ$ on the right hand side denotes matrix multiplication. Theorem 26 holds in this greater generality - a function from $\mathbb{R}^m$ to $\mathbb{R}^n$ is differentiable at a point x_0 if all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ $1 \leq i \leq n$, $1 \leq j \leq m$, are continuous in a neighborhood of x_0 (define a neighborhood of x_0 in $\mathbb{R}^m$!).

Higher derivatives

Just as we repeatedly differentiated a function of one variable to get higher derivatives, we can also look at higher partial derivatives.

However, we now have more freedom. If we have a function $f(x_1, x_2)$ of two variables, we could first take the partial derivative with respect to x_1, then with respect to x_2, then again with respect to x_2, and so on. Does the order in which we differentiate matter?

Theorem 28: Let $f : U \to R$ be a function such that the partial derivatives $\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(f)\right)$ exist and are continuous for every $1 \leq i, j \leq m$. Then,

$$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(f)\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}(f)\right).$$

Functions $f : U \to \mathbb{R}$ for which the mixed partial derivatives of order 2 (that is, the $\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}(f)\right)$) are all continuous are called $\mathcal{C}^2$ functions. Theorem 28 says that for $\mathcal{C}^2$ functions, the order in which one takes partial derivatives does not matter.

From now on we will use use the following notation. By

$$\frac{\partial^n f}{\partial x_1^{n_1} \partial x_2^{n_2} \dots \partial x_k^{n_k}},$$

we mean: first take the partial derivative of f n_1 times with respect to x_1, then n_2 times with respect to x_2, and so on. The number n is nothing but $n_1 + n_2 + \ldots + n_k$. It is called the order of the mixed partial derivative.

Finally, we say that a function is $\mathcal{C}^k$ if all mixed partial derivatives of order k exist and are continuous. A function $f : U \to \mathbb{R}^n$ will be said to be $\mathcal{C}^k$ if each of the functions $f_1, f_2, \ldots, f_n$ are $\mathcal{C}^k$ functions.

From the preceding slide we see the we can talk about $\mathcal{C}^k$ functions for any function from (a subset of) $\mathbb{R}^m$ to $\mathbb{R}^n$. As in the one variable case we can also talk of smooth functions - these are functions for which all partial derivatives of all orders exist. In particular, the notion of a smooth vector field makes perfect sense. There are many interesting facts about smooth vector fields. I will mention just one:
You cannot comb a porcupine.

Or, in more mathematical terms, every smooth tangential vector field on the sphere will vanish at at least one point.

Note that we require that at each point on the sphere the vector we assign must lie in the plane tangent to the sphere at that point.
https://en.wikipedia.org/wiki/Hairy_ball_theorem