MA 109 D1&D2 Lecture 1

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Sequences

Sequences

Definition: A sequence in a set X is a function $a : \mathbb{N} \to X$, that is, a function from the natural numbers to X.

In this course X will usually be a subset of (or equal to) $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$, though we will also have occassion to consider sequences of functions sometimes. In later mathematics courses X may be the complex numbers $\mathbb{C}$ (MA 205), vector spaces (whatever those maybe) the set of continuous functions on an interval $\mathcal{C}([a, b])$ or other sets of functions (MA 106, MA 108, MA 207, MA 214).

Rather than write the value of the function at n as $a(n)$, we often write a_n for the members of the sequence. A sequence is often specified by listing the first few terms

$$a_1, a_2, a_3, \ldots$$

or, more generally by describing the n^{th} term a_n. When we want to talk about the sequence as a whole we sometimes write $\{a_n\}_{n=1}^{\infty}$, but more often we once again just write a_n.

Examples of sequences

1. $a_n = n$ (here we can take $X = \mathbb{N} \subset \mathbb{R}$ if we want, and the sequence is just the identity function. Of course, we can also take $X = \mathbb{R}$).
2. $a_n = 1/n$ (here we can take $X = \mathbb{Q} \subset \mathbb{R}$ if we want, where $\mathbb{Q}$ denotes the rational numbers, or we can take $X = \mathbb{R}$ itself).
3. $a_n = \frac{n!}{n^n}$ ($X = \mathbb{Q}$ or $X = \mathbb{R}$).
4. $a_n = n^{1/n}$ (here the values taken by a_n are irrational numbers, so it best to take $X = \mathbb{R}$).
5. $a_n = \sin\left(\frac{1}{n}\right)$ (again the values taken by a_n are irrational numbers, so it best to take $X = \mathbb{R}$).

These are all examples of sequence of real numbers.

More examples

6. $a_n = \left(n^2, \frac{1}{n}\right)$ (here $X = \mathbb{R}^2$ or $X = \mathbb{Q}^2$).

   This is a sequence in $\mathbb{R}^2$.

7. $f_n(x) = \cos(nx)$ (here X is the set of continuous functions on any interval $[a, b]$ or even on $\mathbb{R}$).

   This is a sequence of functions. More precisely, it is a sequence of continuous functions.

Series

Given a sequence a_n of real numbers, we can manufacture a new sequence, namely its sequence of partial sums:

$$s_1 = a_1, \ s_2 = a_1 + a_2, \ s_3 = a_1 + a_2 + a_3, \ldots$$

More precisely, we have the sequence

$$s_n = \sum_{k=1}^{n} a_k.$$

8. We can take $a_n = r^n$, for some r, i.e., a geometric progression. Then $s_n = \sum_{k=0}^{n} r^k$.

9. $s_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$, or writing it out $s_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$.

   We get a sequence of polynomial functions.

Monotonic sequences

For the moment we will concentrate on sequences in $\mathbb{R}$.

Definition: A sequence is said to be a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition: A sequence is said to be a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A monotonic sequence is one that is either monotonically increasing or monotonically decreasing.

From the examples in the previous slide, Example 1 is a monotonically increasing sequence, Example 2 is a monotonically decreasing sequence.
How about Example 3?

In Example 3 we notice that if $a_n = \frac{n!}{n^n}$,

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}} = a_n \times \frac{(n+1)n^n}{(n+1)^{(n+1)}} \leq a_n,$$

so the sequence is monotonically decreasing.

Eventually monotonic sequences

In Example 4 ($a_n = n^{1/n}$), we note that

$$a_1 = 1 < 2^{1/2} = a_2 < 3^{1/3} = a_3,$$

(raise both a_2 and a_3 to the sixth power to see that $2^3 < 3^2$!). However, $3^1/3 > 4^{1/4} > 5^{1/5}$. So what do you think happens as n gets larger?

In fact, $a_{n+1} \leq a_n$, for all $n \geq 3$. Prove this fact as an exercise. Such a sequence is called an eventually monotonic sequence, that is, the sequence becomes monotonic(ally decreasing) after some stage. One can similarly define eventually monotonically increasing sequences.

Let us quickly run through the other examples. Example 5 - monotonically decreasing. Example 6 - is not a sequence of real numbers. Example 7 - is a sequence of real numbers if we fix a value of x . Can it be monotonic for some x? Example 8 is monotonic for any fixed value of r and so is Example 9 for any non-negative value of x.