MA 109 D1&D2 Lecture 3

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Sequences: recap

Limits of sequences

Sequences

Definition: A sequence in a set X is a function $a : \mathbb{N} \to X$, that is, a function from the natural numbers to X.

Sometimes we will also call a function $a : \mathbb{W} \to X$, a sequence in X. This just means that we start counting from 0 rather than 1.

Given a sequence a_n of real numbers, we can manufacture a new sequence, namely its sequence of partial sums:

$$s_1 = a_1, \; s_2 = a_1 + a_2, \; s_3 = a_1 + a_2 + a_3, \ldots$$

More precisely, we have the sequence

$$s_n = \sum_{k=1}^{n} a_k.$$

Monotonic sequences

For the moment we will concentrate on sequences in $\mathbb{R}$.

Definition: A sequence is said to be a monotonically increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ (Examples: $a_n = n$ or $a_n = 1 - 1/n$).

Definition: A sequence is said to be a monotonically decreasing sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ (Examples: $a_n = -n$ or $a_n = 1/n$).

A monotonic sequence is one that is either monotonically increasing or monotonically decreasing.

A sequence is called eventually monotonically increasing if there exists $N \in \mathbb{N}$ such that $a_{n+1} \geq a_n$ for all $n > N$. (Example: a sequence a_n defined as $a_1 = 10, a_2 = 1, a_n = n$ if $n \geq 3$).

A sequence is called eventually monotonically decreasing if there exists $N \in \mathbb{N}$ such that $a_{n+1} \leq a_n$ for all $n > N$. (Example: $a_n = n^{1/n}$).

Limits: Preliminaries

While all of you are familiar with limits, most of you have probably not worked with a rigourous definition.

So what does it mean for a sequence to tend to a limit? Let us look at the sequence $a_n = 1/n^2$. We wish to study the behaviour of this sequence as n gets large. Clearly as n gets larger and larger, $1/n^2$ gets smaller and smaller and seems to approach the value 0, or more precisely

the distance between $1/n^2$ and 0 becomes smaller and smaller.

In fact (and this is the key point), by choosing n large enough, we can make the distance between $1/n^2$ and 0 smaller than any prescribed quantity.

Let us examine the above statement, and then try and quantify it.

More precisely:

The distance between $1/n^2$ and 0 is given by $|1/n^2 - 0| = 1/n^2$.

Suppose I require that $1/n^2$ be less that 0.1 (that is 0.1 is my prescribed quantity). Clearly, $1/n^2 < 1/10$ for all $n > 3$.

Similarly, if I require that $1/n^2$ be less than $0.0001 (= 10^{-4})$, this will be true for all $n > 100$.

We can do this for any number, no matter how small. If $\epsilon > 0$ is any number,

$$1/n^2 < \epsilon \iff 1/\epsilon < n^2 \iff n > 1/\sqrt{\epsilon}.$$

In other words, given any $\epsilon > 0$, we can always find a natural number N (in this case any $N > 1/\sqrt{\epsilon}$) such that for all $n > N$, $|1/n^2 - 0| < \epsilon$.

The rigourous definition of a limit

Motivated by the previous example, we define the limit as follows.

Definition: A sequence a_n tends to a limit l/converges to a limit l, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon$$

whenever $n > N$.

This is what we mean when we write

$$\lim_{n \to \infty} a_n = l.$$

If we just want to say that the sequence has a limit without specifying what that limit is, we simply say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges, or that it is convergent.
A sequence that does not converge is said to diverge, or to be divergent.

Remarks on the definition

Remarks

1. Note that the N will (of course) depend on ϵ, as it did in our example, so it would have been more correct to write $N(\epsilon)$ in the definition of the limit. However, we usually omit this extra bit of notation.
2. We have already shown that $\lim_{n\to\infty} 1/n^2 = 0$. The same argument works for $\lim_{n\to\infty} 1/n^{\alpha}$, for any real $\alpha > 0$. We just take N to be any integer bigger than $1/\epsilon^{1/\alpha}$ for a given ϵ.
3. For a given ϵ, once one N works, any larger N will also work. In order to show that a sequence tends to a limit l we are not obliged to find the best possible N for a given ϵ, just some N that works. Thus, for the sequence $1/n^2$ and $\epsilon = 0.1$, we took $N = 3$, but we can also take $N = 10, 100, 1729$, or any other number bigger than 3.
4. Showing that a sequence converges to a limit l is not easy. One first has to guess the value l and then prove that l satisfies the definition. We will see how to get around this in various ways.

More examples of limits

Let us show that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$.

For this we note that for $x \in [0, \pi/2]$, $0 \leq \sin x \leq x$ (try to remember why this is true).
Hence,

$$|\sin 1/n - 0| = |\sin 1/n| < 1/n.$$

Thus, given any $\epsilon > 0$, if we choose some $N > 1/\epsilon$, $n > N$ implies $1/n < 1/N < \epsilon$. It follows that $|\sin 1/n - 0| < \epsilon$.

Let us consider Exercise 1.1.(ii) of the tutorial sheet. Here we have to show that $\lim_{n\to\infty\infty} 5/(3n+1) = 0$. Once again, we have only to note that

$$\frac{5}{3n+1} < \frac{5}{3n},$$

and if this is to be smaller than ϵ, we must have $n > N > 5/3\epsilon$.

Formulæ for limits

If a_n and b_n are two convergent sequences then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$, provided $\lim_{n\to\infty} b_n \neq 0$

Implicit in the formulæ is the fact that the limits on left hand side exist.

Note that the constant sequence $a_n = c$ has limit c, so as a special case of (2) above we have

$$\lim_{n\to\infty}(c \cdot b_n) = c \cdot \lim_{n\to\infty} b_n.$$

Using the formulæ above we can break down the limits of more complicated sequences into simpler ones and evaluate them.

The Sandwich Theorem(s)

Theorem 1: If a_n, b_n and c_n are convergent sequences such that $a_n \le b_n \le c_n$ for all n, then

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n \le \lim_{n\to\infty} c_n.$$

A second version of the theorem is especially useful:
Theorem 2: Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$. If b_n is a sequence satisfying $a_n \le b_n \le c_n$ for all n, then b_n converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Note that we do not assume that b_n converges in this version of the theorem - we get the convergence of b_n for free . Together with the rules for sums, differences, products and quotients, this theorem allows us to handle a large number of more complicated limits.

An example using the theorems above

Consider Exercise 1.2.(iii) on the tutorial sheet. We have to show that

$$\lim_{n\to\infty} \frac{n^3+3n^2+1}{n^4+8n^2+2}$$

exists and to evaluate it.

It is clear that

$$0 < \frac{n^3+3n^2+1}{n^4+8n^2+2} \leq \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}.$$

(How do we get this?)
Note that $n^3/(n^4+8n^2+2) < n^3/n^4 = 1/n$, and the other two terms can be handled similarly.)

Hence, applying the Sandwich Theorem (Theorem 2) to the sequences

$$a_n = 0, \quad b_n = \frac{n^3+3n^2+1}{n^4+8n^2+2} \quad \text{and} \quad c_n = \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^4}$$

we see that the limit we want exists provided $\lim_{n\to\infty} c_n$ exists, so this is what we must concentrate on proving.

The limit $\lim_{n\to\infty} c_n$ exists provided each of the terms appearing in the sum has a limit and in that case it is equal to the sum of the limits (by the first formula). But each of these limits is quite easy to evaluate.

We already know that

$$\lim_{n\to\infty} 1/n = 0 = \lim_{n\to\infty} 1/n^4,$$

while

$$\lim_{n\to\infty} 3/n^2 = 3 \cdot \lim_{n\to\infty} 1/n^2 = 0$$

where we have used the special case of the second formula (limit of the product is the product of the limits) for the first equality in the equation above. Since all three limits converge to 0, it follows the given limit is $0 + 0 + 0 = 0$.

Bounded Sequences

The formulæ and theorems stated above can be easily proved starting from the definitions. We will prove the second formula and leave the other proofs as exercises.
Definition: A sequence a_n is said to be bounded if there is a real number $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$. A sequence that is not bounded is called unbounded.

In our list of examples, Example 1 ($a_n = n$) is an example of an unbounded sequence, while Examples 2 - 5 ($a_n = 1/n, \sin(1/n), n!/n^n, n^{1/n}$) are examples of bounded sequences.

Bounded sequence don't necessarily converge - for instance $a_n = (-1)^n$. However,

Convergent sequences are bounded

Lemma: Every convergent sequence is bounded.

Proof: Suppose a_n converges to l. Choose $\epsilon = 1$. There exists $N \in \mathbb{N}$ such that $|a_n - l| < 1$ for all $n > N$. In other words, $l - 1 < a_n < l + 1$, for all $n > N$, which gives $|a_n| < |l| + 1$ for all $n > N$. Let

$$M_1 = \max\{|a_1|, |a_2|, \ldots, |a_N|\}$$

and let $M = \max\{M_1, |l| + 1\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. $\square$

In the slides presented in class, I had forgotten to put absolute value signs in many places in the proof above and in the next slide. This has now been corrected.

We will use this Lemma to prove the product rule for limits.

The proof of the product rule

We wish to prove that $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

(This proof works for $l_2 \neq 0$. What happens for $l_2 = 0$?)
Suppose $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} b_n = l_2$. We need to show that $\lim_{n\to\infty} a_n b_n = l_1 l_2$.

Fix $\epsilon > 0$. We need to show that we can find $N \in \mathbb{N}$ such that $|a_n b_n - l_1 l_2| < \epsilon$, whenever $n > N$. Notice that

$$\begin{aligned}
|a_n b_n - l_1 l_2| &= |a_n b_n - a_n l_2 + a_n l_2 - l_1 l_2| \\
&= |a_n(b_n - l_2) + (a_n - l_1) l_2| \\
&\leq |a_n||b_n - l_2| + |a_n - l_1||l_2|,
\end{aligned}$$

where the last inequality follows from the triangle inequality. So in order to guarantee that the left hand side is small, we must ensure that the two terms on the right hand side together add up to less than ϵ. In fact, we make sure that each term is less than $\epsilon/2$.

The proof of the product rule, continued

Since a_n is convergent, it is bounded by the lemma we have just proved. Hence, there is an M such that $|a_n| < M$ for all $n \in \mathbb{N}$.

Given the quantities $\epsilon/2|l_2|$ and $\epsilon/2M$, there exist N_1 and N_2 such that

$$|a_n - l_1| < \epsilon/2|l_2| \quad \text{and} \quad |b_n - l_2| < \epsilon/2M.$$

Let $N = \max\{N_1, N_2\}$. If $n > N$, then both the inequalities above hold. Hence, we have

$$|a_n||b_n - l_2| < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2} \quad \text{and} \quad |a_n - l_1||l_2| < |l_2| \cdot \frac{\epsilon}{2|l_2|} = \frac{\epsilon}{2}.$$

Now it follows that

$$|a_n b_n - l_1 l_2| \le |a_n||b_n - l_2| + |a_n - l_1||l_2| < \epsilon,$$

for all $n > N$, which is what we needed to prove. □

The proofs of the other rules for limits are similar to the one we proved above. Try them as exercises.

A guarantee for convergence

As we mentioned earlier, proving that a limit exists is hard because we have to guess what its value might be and then prove that it satisfies the definition. The following theorem guarantees the convergence of a sequence without knowing the limit beforehand.

Definition: A sequence a_n is said to be bounded above (resp. bounded below) if $a_n < M$ (resp. $a_n > M$) for some $M \in \mathbb{R}$.
A sequence that is bounded both above and below is obviously bounded.

Theorem 3: A montonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges.

Remarks on Theorem 3

Theorem 3 clearly makes things very simple in many cases. For instance, if we have a monotonically decreasing sequence of positive numbers, it must have a limit, since 0 is always a lower bound!

Can we guess what the limit of a monotonically increasing sequence a_n bounded above might be?
It will be the supremum or least upper bound (lub) of the sequence. This is the number, say M which has the following properties:

1. $a_n \leq M$ for all n and
2. If M_1 is such that $a_n < M_1$ for all n, then $M \leq M_1$.

The point is that a sequence bounded above may not have a maximum but will always have a supremum. As an example, take the sequence $1 - 1/n$. Clearly there is no maximal element in the sequence, but 1 is its supremum.