MA 105 D3 Lecture 13

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The fundamental theorem of calculus

An example of a function that is not Darboux integrable

Here is a function that is not Darboux integrable of $[0,1]$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

It should be clear that no matter what partition one takes the infimum on any sub-interval in the partition will be 0 and the supremum will be 1.

From this one can see immediately that

$$L(f,P) = 0 \neq 1 = U(f,P),$$

for every P, and hence that $L(f) = 0 \neq 1 = U(f)$.

Another property of the Riemann Integral

Theorem 23: Suppose f is Riemann integrable on $[a, b]$ and $c \in [a, b]$. Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proof: First we note that if $c = a$ or $c = b$, there is nothing to prove.

Next, if $c \in (a, b)$ we proceed as follows. If P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$, then $P_1 \cup P_2 = P'$ is obviously a partition of $[a, b]$. Thus, partitions of the form $P_1 \cup P_2$ constitute a subset of the set of all partitions of $[a, b]$. For such partitions P' we have

$$L(f, P') = L(f, P_1) + L(f, P_2).$$

Let us denote by $L(f)_{[a,c]}$ (resp. $L(f)_{[c,b]}$) the Darboux lower integral of f on the interval $[a, c]$ (resp. $[c, b]$).

If we take the supremum over all partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ we get

$$\sup_{P'} L(f, P') = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

Now the supremum on the left hand side is taken only over partitions P' having the special form $P_1 \cup P_2$. Hence it is less than or equal to $\sup_P L(f, P)$ where this supremum is taken over all partitions P. We thus obtain

$$L(f)_{[a,c]} + L(f)_{[c,b]} \leq L(f).$$

On the other hand, for any partition $P = \{a < x_1 < \dots x_{n-1} < b\}$ we can consider the partition $P' = P \cup \{c\}$. This will be a refinement of the partition P and can be written as a union of two partitions P_1 of $[a, c]$ and P_2 of $[c, b]$.

By the property for refinements for Darboux sums we know that $L(f, P) \leq L(f, P')$.

Thus, given any partition P of $[a, b]$, there is a refinement P' which can be written as the union of two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively, and by the above inequality,

$$\sup_P L(f, P) \leq \sup_{P'} L(f, P'),$$

where the first supremum is taken over all partitions of $[a, b]$ and the second only over those partitions P' which can be written as a union of two partitions as above. This shows that

$$L(f) \leq L(f)_{[a,c]} + L(f)_{[c,b]},$$

so, together with the previous inequality, we get

$$L(f) = L(f)_{[a,c]} + L(f)_{[c,b]}.$$

The same kind of reasoning applies to the Upper sums which allows us to prove the required property.

Motivation

The Fundamental Theorem of calculus allows us to relate the process of Riemann integration to the process of differentiation. Essentially, it tells us that integrating and differentiating are inverse processes. This is a tremendously useful theorem for several reasons.

It turns out that (Riemann) integrating even simple functions is much harder than differentiating them (if you don't believe me, try integrating $(\tan x)^3$ via Riemann sums!). In practice, however, integration is what we need to do to solve physical problems. Usually, when we are studying the motion of a particle or a planet what we find is that the position of a particle, which is a function of time, satisfies some differential equation. Solving the differential equation involves performing the inverse operation of taking some combination of derivatives. The simplest such inverse operation is taking the inverse of the first derivative, which the Fundamental Theorem says, is the same as integrating.

Calculating Integrals

Thus, calculating integrals is one of the basic things one needs to do for solving even the simplest physics and engineering problems. The problem is that this is quite difficult to do.

Once we know the derivatives of some basic functions (polynomials, trigonometric functions, exponentials, logarithms) we can differentiate a wide class of functions using the rules for differentiation , especially the product and chain rules. By contrast, the only rule for Riemann integration that can be proved from the basic definitions is the sum rule.

The Fundamental Theorem solves this problem (partially) because it allows us to deduce formalæ for the integrals of the products and the composition of functions from the corresponding rules for derivatives.

The Fundamental Theorem - Part I

Theorem 24: Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let

$$F(x) = \int_a^x f(t)dt$$

for any $x \in [a, b]$. Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x),$$

for all $x \in (a, b)$.

Proof: We know that $f(t)$ is Riemann integrable for any $x \in [a, b]$ because of Theorem 21 (every continuous function is Riemann integrable).

The proof of Part I continued

By Theorem 23 we know that

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt,$$

for $x + h \in [a, b]$. Hence

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f(t)dt.$$

We know that if $f(t) \leq g(t)$ on $[a, b]$, then $\int f(t)dt \leq \int g(t)dt$. We apply this to the three functions $m(h)$, f and $M(h)$, where $m(h)$ and $M(h)$ are the constant functions given by the minimum and maximum of the function f on $[x, x + h]$ to get:

$$m(h) \cdot h \leq \int_x^{x+h} f(t)dt \leq M(h) \cdot h.$$

Dividing by h and taking the limit gives

$$\lim_{h \to 0} m(h) \leq \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \leq \lim_{h \to 0} M(h).$$

But f is a continuous function, so $\lim_{h\to 0} m(h) = \lim_{h\to 0} M(h) = f(x)$. By the Sandwich theorem for limits (use version 2), we see that limit in the middle exists and is equal to $f(x)$, that is $F'(x) = f(x)$. This proves the first part of the Fundamental Theorem of Calculus. $\square$

This first form of the Fundamental Theorem allows us to compute definite integrals. Keeping the notation as in the Theorem we obtain

Corollary:

$$\int_c^d f(t)dt = F(d) - F(c),$$

for any two points $c, d \in [a, b]$.

The Fundamental Theorem of Calculus Part 2

Theorem 24: Let $f : [a, b] \to \mathbb{R}$ be given and suppose there exists a continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) and which satisfies $g'(x) = f(x)$. Then, if f is Riemann integrable on $[a, b]$,

$$\int_a^b f(t)dt = g(b) - g(a).$$

Note that this statement does not assume that the function $f(t)$ is continuous, and is thus stronger than the corollary just stated.

Proof: We can write:

$$g(b) - g(a) = \sum_{i=1}^{n} [g(x_i) - g(x_{i-1})],$$

where $\{a = x_0, x_1, \ldots, x_n = b\}$ is an arbitrary partition of $[a, b]$. Using the mean value theorem for each of the intervals $I_j = [x_j, x_{j-1}]$, we can write

The proof of the Fundamental Theorem part II continued

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}).$$

where $c_i \in (x_{i-1}, x_i)$.

Substituting this in the previous expression and using the fact that $g'(c_i) = f(c_i)$, we get

$$g(b) - g(a) = \sum_{i=1}^{n} [f(c_i)(x_i - x_{i-1})].$$

The calculation above is valid for any partition The right hand side obviously represents a Riemann sum. By hypothesis f is Riemann integrable. It follows (using Definition 1, for example) that as $\|P\| \to 0$, the right hand side goes to the Riemann integral. $\square$

The formula for arc length

Let us denote the arc length of the curve $y = f(x)$ by S. The length of any given hypotenuse in the previous slide is given by the Pythagorean Theorem: $\sqrt{\Delta x^2 + \Delta y^2}$.
Intuitively, the sum of the lengths of the n hypotenuses appears to approximate S:

$$S \sim \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i,$$

where "$\sim$" means approximately equal. We can use this idea to define the arc length as

$$S := \lim_{\Delta x_i \to 0} \sum_{i=1}^{\infty} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

provided this limit exists (in particular, we demand that the limit is a finite number).

Exercise 4.10.(ii) Find the length of the curve

$$y(x) = \int_0^x \sqrt{\cos 2t}\, dt,\ 0 \leq x \leq \pi/4.$$

Solution: The formula for the arc length of a curve $y = f(x)$ between the points $x = a$ and $x = b$ is given by

$$\int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

For the problem at hand this gives

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x} dx = \sqrt{2} \int_0^{\pi/4} \cos(x) dx = 1.$$

Rectifiable curves

Not all curves have finite arc length! Here is an example of a curve with infinite arc length.

Example: Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the curve given by $\gamma(t) = (t, f(t))$, where

$$f(t) = \begin{cases} t\cos\left(\frac{\pi}{2t}\right), \text{ if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$$

If

http://math.stackexchange.com/questions/296397/nonrectifiable-curve

is correct, you should be able to check that this curve has infinite arc length. Try it as an exercise.

Notice that the curve above is given by a continuous function. Curves for which the arc length S is finite are called rectifiable curves. You can easily check that the graphs of piecewise $\mathcal{C}^1$ functions are rectifiable.

Things can get even stranger

In fact, there exist space filling curves, that is curves $\gamma : [0,1] \to [0,1] \times [0,1]$ which are continuous and surjective. Obviously the graph of this curve "fills up" the entire square. Such curves are not rectifiable (can you prove this?)

The existence of such curves should make you question whether your intuitive notion of dimension actually has any mathematical basis. If a line segment can be mapped continuously onto a square, is it reasonable to say that they have different dimensions? After all, this means we can describe any point on the square using just one number.

We will answer this question (without a proof) later in this course. We will also come back to arc length of a curve when studying multivariable calculus.