MA 105 D3 Lecture 18

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Functions from $\mathbb{R}^m \to \mathbb{R}^n$

So far we have only studied functions whose range was a subset of $\mathbb{R}$. Let us now allow the range to be $\mathbb{R}^n$, $n = 1, 2, 3, \ldots$. Can we understand what continuity, differentiability etc. mean?

Let U be a subset of $\mathbb{R}^m$ ($m = 1, 2, 3, \ldots$) and let $f : U \to \mathbb{R}^n$ be a function. If $x = (x_1, x_2, \ldots, x_m) \in U$, $f(x)$ will be an n-tuple where each coordinate is a function of x. Thus, we can write $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where each $f_i(x)$ is a function from U to $\mathbb{R}$.

Functions which take values in $\mathbb{R}$ are called scalar valued functions, which functions which take values in $\mathbb{R}^n$, $n > 1$ are usually called vector valued functions.

Continuity of vector valued functions

The definition of continuity is exactly the same as before.
Definition: The function f is said to be continuous at a point $c \in U$ if

$$\lim_{x \to c} f(x) = f(c).$$

How does one define the limit on the left hand side? The function

f takes values in $\mathbb{R}^n$, so its limit must be a point in $\mathbb{R}^n$, say $l = (l_1, l_2, \ldots, l_n)$.

Definition: We say that $f(x)$ tends to the limit l if given any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \|x - c\|_m < \delta$, then

$$\|f(x) - l\|_n < \epsilon.$$

You can easily prove the following theorem yourself:

Theorem: The function $f : U \to \mathbb{R}^n$ is continuous if and only if each of the functions $f_i : U \to \mathbb{R}$, $1 \leq i \leq n$, is continuous.

Vector fields

When $m = n$, vector valued functions are often called vector fields. We will study vector fields in slightly greater detail when $m = n = 2$ and $m = n = 3$.

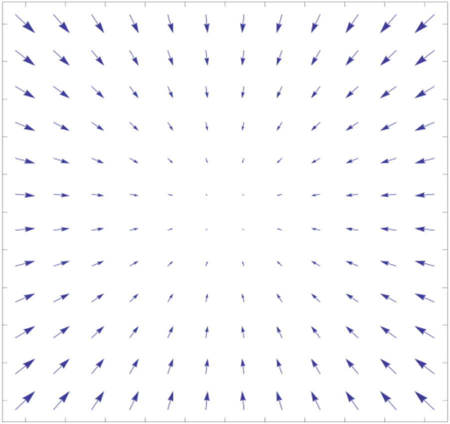
When $n = 1$, we get scalar valued functions and these are called scalar fields. The function which assigns the temperature to a point in space is an example of a scalar field.

We have already seen one example of a vector field - the gravitational force field $-\frac{GMm}{r^3} \cdot \mathbf{r}$ felt by a mass m whose position vector with respect to a mass M at the origin is $\mathbf{r}$. In this particular case we showed the the force field arose as the gradient of a scalar valued function (the potential $V = GMm/r$).

One of the most important questions in calculus is the following: Given a vector field, when does it arise as the gradient of a scalar function? In physics, vector force fields that arise from a scalar potential function are called conservative.

Some pictures of vector fields

We can actually visualize two dimensional vector fields as follows. At each point in $\mathbb{R}^2$ we can draw an arrow starting at that point pointing in the direction of the image vector and with size proportional to the magnitude of the image vector.



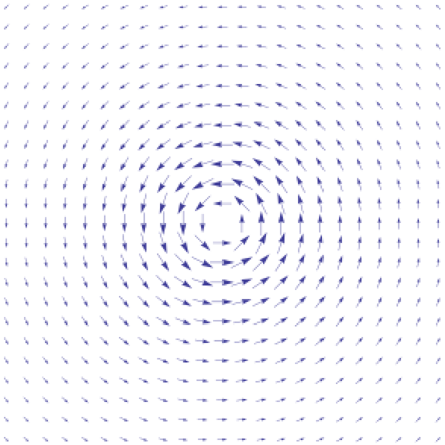
What function from $\mathbb{R}^2$ to $\mathbb{R}^2$ does this picture represent?

$$f(x, y) = (-x, -y)$$

the the radial vector field.

http://en.wikipedia.org/wiki/File:Radial_vector_field_dense.svg

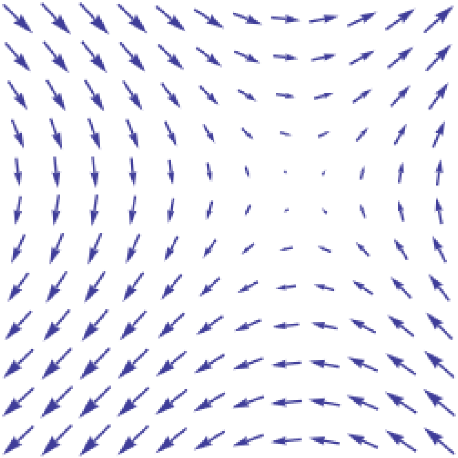
How about this one?



$$f(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

This is an example of an irrotational vector field.
It cannot be written as the gradient of a potential function.

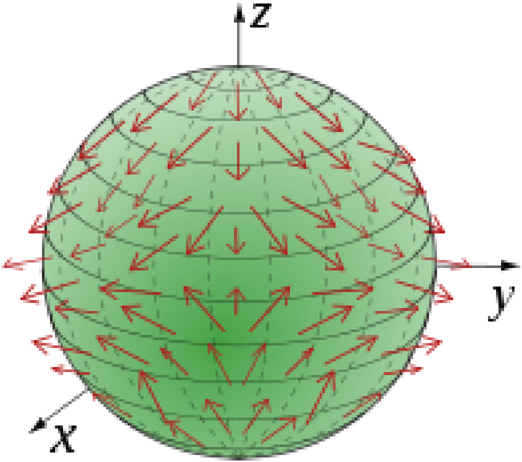
Here is another (more complicated one)



$$f(x, y) = (\sin y, \sin x)$$

http://en.wikipedia.org/wiki/File:VectorField.svg

One can also talk about two dimensional vector fields on any two dimensional surface. Here is a picture of a vector field on a sphere.



http://en.wikipedia.org/wiki/File:Vector_sphere.svg

Vector fields in the real world

Many real world phenomena can be understood using the language of vector fields. In physics, apart from gravitation, electromagnetic forces can also be represented by vector fields. That is, to each point in space we attach the vector representing the force at that point. Such fields are called force fields.

Fluids flowing are also often modeled using vector fields, with each point being mapped to the vector representing the velocity of the fluid flow. For instance, the velocity of winds in the atmosphere can be represented as a vector field. Such fields are called velocity fields.

The derivative for $f : U \to \mathbb{R}^n$

We now define the derivative for a function $f : U \to \mathbb{R}^n$, where U is a subset of $\mathbb{R}^m$.

The function f is said to be differentiable at a point x if there exists a an $n \times m$ matrix $Df(x)$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x+h) - f(x) - Df(x) \cdot h\|}{\|h\|} = 0.$$

Here $x = (x_1, x_2, \ldots, x_m)$ and $h = (h_1, h_2, \ldots h_m)$ are vectors in $\mathbb{R}^m$.

The matrix $Df(x)$ is usually called the total derivative of f. It is also referred to as the Jacobian matrix. What are its entries?

From our experience in the 2×1 case we might guess (correctly!) that the entries will be the partial derivatives.

Here is the total derivative or the derivative matrix written out fully.

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_m}(x) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & \cdots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix}$$

In the 2×2 case we get

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{pmatrix}.$$

As before, the derivative may be viewed as a linear map, this time from $\mathbb{R}^m$ to $\mathbb{R}^n$ (or, in the case just above, from $\mathbb{R}^2$ to $\mathbb{R}^2$).

Rules for the total derivative

Just like in the one variable case, it is easy to prove that

$$D(f+g)(x) = Df(x) + Dg(x).$$

Somewhat harder, but only because the notation gets more cumbersome, is the Chain rule:

$$D(f \circ g)(x) = Df(g(x)) \circ Dg(x),$$

where $\circ$ on the right hand side denotes matrix multiplication. Theorem 26 holds in this greater generality - a function from $\mathbb{R}^m$ to $\mathbb{R}^n$ is differentiable at a point x_0 if all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ $1 \leq i \leq n$, $1 \leq j \leq m$, are continuous in a neighborhood of x_0 (define a neighborhood of x_0 in $\mathbb{R}^m$!).

Review - problems involving the gradient

Exercise 1: Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$.

Solution: The hyperboloid is an implicitly defined surface. A normal vector at a point (x_0, y_0, z_0) on the hyperboloid is given by the gradient of the function $x^2 - y^2 + 2z^2$ at (x_0, y_0, z_0):

$$\nabla f(x_0, y_0, z_0) = (2x_0, -2y_0, 4z_0).$$

We require this vector to be parallel to the line joining the points $(3, -1, 0)$ and $(5, 3, 6)$. This line lies in the same direction as the vector $(5 - 3, 3 + 1, 6 - 0) = (2, 4, 6)$. Thus we need only solve the equations

$$2x_0 = 2, \quad -2y_0 = 4, \quad 4z_0 = 6,$$

which give $x_0 = 1$, $y_0 = -2$ and $z_0 = 3/2$. Thus, we need to find λ such that $\lambda(1, -2, 3/2)$ lies on the hyperboloid. Substituting in the equation yields $\lambda = \pm\sqrt{2/3}$.

Problems involving the gradient, continued

Exercise 2: Find the directions in which the directional derivative of $f(x,y) = x^2 + \sin xy$ at the point $(1,0)$ has the value 1.

Solution: We compute ∇f first:

$$\nabla f(x,y) = (2x + y\cos xy, x\cos xy),$$

so at $(1,0)$ we get, $\nabla f(1,0) = (2,1)$.

To find the directional derivative in the direction $v = (v_1, v_2)$ (where v is a unit vector), we simply take the dot product with the gradient:

$$\nabla_v f(1,0) = 2v_1 + v_2.$$

This will have value "1" when $2v_1 + v_2 = 1$, subject to $v_1^2 + v_2^2 = 1$, which yields $v_1 = 0, v_2 = 1$ or $v_1 = 4/5, v_2 = -3/5$.