

# MA 105 D3 Lecture 3

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## Tutorial problems for July 31

The numbers refer to the tutorial sheet. E.g. 1.1 (iii) means Problem no. 1 part (iii) of the first tutorial sheet.

1.1(iii), 1.2(iv), 1.3(i), 1.5(iii), 1.7 and 1.9.

## Formulæ for limits

If  $a_n$  and  $b_n$  are two convergent sequences then

1.  $\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$
2.  $\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n.$
3.  $\lim_{n \rightarrow \infty} (a_n / b_n) = \lim_{n \rightarrow \infty} a_n / \lim_{n \rightarrow \infty} b_n,$  provided  $\lim_{n \rightarrow \infty} b_n \neq 0$

Implicit in the formulæ is the fact that the limits on left hand side exist.

Note that the constant sequence  $a_n = c$  has limit  $c$ , so as a special case of (2) above we have

$$\lim_{n \rightarrow \infty} (c \cdot b_n) = c \cdot \lim_{n \rightarrow \infty} b_n.$$

Using the formulæ above we can break down the limits of more complicated sequences into simpler ones and evaluate them.

# The Sandwich Theorem(s)

**Theorem 1:** If  $a_n$ ,  $b_n$  and  $c_n$  are convergent sequences such that  $a_n \leq b_n \leq c_n$  for all  $n$ , then

$$\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n \leq \lim_{n \rightarrow \infty} c_n.$$

A second version of the theorem is especially useful:

**Theorem 2:** Suppose  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n$ . If  $b_n$  is a sequence satisfying  $a_n \leq b_n \leq c_n$  for all  $n$ , then  $b_n$  converges and

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n.$$

Note that we **do not assume that  $b_n$  converges in this version of the theorem - we get the convergence of  $b_n$  for free** . Together with the rules for sums, differences, products and quotients, this theorem allows us to handle a large number of more complicated limits.

# Bounded Sequences

**Definition:** A sequence  $a_n$  is said to be **bounded** if there is a real number  $M > 0$  such that  $|a_n| \leq M$  for every  $n \in \mathbb{N}$ . A sequence that is not bounded is called **unbounded**.

**Lemma:** Every convergent sequence is bounded.

## A guarantee for convergence

As we mentioned earlier, proving that a limit exists is hard because we have to guess what its value might be and then prove that it satisfies the definition. The following theorem guarantees the convergence of a sequence without knowing the limit beforehand.

**Definition:** A sequence  $a_n$  is said to be **bounded above** (resp. **bounded below**) if  $a_n < M$  (resp.  $a_n > M$ ) for some  $M \in \mathbb{R}$ .

A sequence that is bounded both above and below is obviously bounded.

**Theorem 3:** A monotonically increasing (resp. decreasing) sequence which is bounded above (resp. below) converges.

## Remarks on Theorem 3

Theorem 3 clearly makes things very simple in many cases. For instance, if we have a monotonically decreasing sequence of positive numbers, it must have a limit, since 0 is always a lower bound!

Can we guess what the limit of a monotonically increasing sequence  $a_n$  bounded above might be?

It will be the **supremum** or **least upper bound (lub)** of the sequence. This is the number, say  $M$  which has the following properties:

1.  $a_n \leq M$  for all  $n$  and
2. If  $M_1$  is such that  $a_n < M_1$  for all  $n$ , then  $M \leq M_1$ .

The point is that a sequence bounded above may not have a maximum but will always have a supremum. As an example, take the sequence  $1 - 1/n$ . Clearly there is no maximal element in the sequence, but 1 is its supremum.

## Another monotonic sequence

Let us look at Exercise 1.5.(i) which considers the sequence

$$a_1 = 3/2 \quad \text{and} \quad a_{n+1} = \frac{1}{2} \left( a_n + \frac{2}{a_n} \right).$$

$$\begin{aligned} a_{n+1} < a_n &\iff \frac{1}{2} \left( a_n + \frac{2}{a_n} \right) < a_n \\ &\iff \sqrt{2} < a_n. \end{aligned}$$

On the other hand,

$$\frac{1}{2} \left( a_n + \frac{2}{a_n} \right) \geq \sqrt{2}, \quad (\text{Why is this true?})$$

so  $a_{n+1} \geq \sqrt{2}$  for all  $n \geq 1$  and  $a_1 > \sqrt{2}$  is given.

Hence,  $\{a_n\}_{n=1}^{\infty}$  is a monotonically decreasing sequence, bounded below by  $\sqrt{2}$ . By Theorem 3, it converges.

**Exercise 1.** What do you think is the limit of the above sequence (Refer to the supplement to Tutorial 1)?

## More remarks on limits

**Exercise 2.** More generally, what is the limit of a monotonically decreasing sequence bounded below? How can you describe it? This number is called the **infimum or greatest lower bound (glb)** of the sequence.

The proof of Theorem 3 is not so easy and more or less involves understanding what a real number is. It is related to the notion of Cauchy sequences about which I will try to say something a little later (again, refer to the supplement to Tutorial 1).

**An important remark:** If we change finitely many terms of a sequence it does not affect the convergence and boundedness properties of a sequence.

If it is convergent, the limit will not change. If it is bounded, it will remain bounded though the supremum may change. Thus, an eventually monotonically increasing sequence bounded above will converge (formulate the analogue for decreasing sequences).

Bottomline: **From the point of view of the limit, only what happens for large  $N$  matters.**

## Cauchy sequences

As we have seen, it is not easy to tell whether a sequence converges or not because we have to first guess what the limit might be, and then try and prove that the sequence actually converges to this limit. For a monotonic sequence, we have a criterion, but what about more general sequences?

There is another very useful notion which allows us to decide whether the sequence converges **by looking only at the elements of the sequence itself**. We describe this below.

**Definition:** A sequence  $a_n$  in  $\mathbb{R}$  is said to be a **Cauchy sequence** if for every  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|a_n - a_m| < \epsilon,$$

for all  $m, n > N$ .

## Cauchy sequences: the theorem

**Theorem 4:** Every Cauchy sequence in  $\mathbb{R}$  converges.

**Remark 1:** One can now check the convergence of a sequence just by looking at the sequence itself!

**Remark 2:** One can easily check the converse:

**Theorem 5:** Every convergent sequence is Cauchy.

**Remark 3:** Remember that when we defined sequences we defined them to be functions from  $\mathbb{N}$  to  $X$ , for any set  $X$ . So far we have only considered  $X = \mathbb{R}$ , but as we said earlier we can take other sets, for instance, subsets of  $\mathbb{R}$ . For instance, if we take  $X = \mathbb{R} \setminus \{0\}$ , Theorem 4 is not valid. The sequence  $1/n$  is a Cauchy sequence in this  $X$  but obviously does not converge in  $X$ . If we take  $X = \mathbb{Q}$ , the example given in 1.5.(i) ( $a_{n+1} = (a_n + 2/a_n)/2$ ) is a Cauchy sequence in  $\mathbb{Q}$  which does not converge in  $\mathbb{Q}$ . Thus Theorem 4 is really a theorem about real numbers.

# The completeness of $\mathbb{R}$

A set in which every Cauchy sequence converges is called a complete set. Thus Theorem 4 is sometimes rewritten as

**Theorem 4'**: The real numbers are complete.

We will see other examples of complete sets, but we can now address (very briefly) the question of what a real number is. More precisely, we can *construct* the set of real numbers starting with the rational numbers.

We let  $S$  be the set of all Cauchy sequences with values in  $\mathbb{Q}$ . We will put a relation on this set.

# The definition of a real number

I did not do this slide in our class but did mention it in the other division

Two sequence  $\{a_n\}$  and  $\{b_n\}$  will be related to each other (and we write  $a_n \sim b_n$ ) if

$$\lim_{n \rightarrow \infty} |a_n - b_n| = 0$$

You can check that this is an equivalence relation and it is a fact that it *partitions* the set  $S$  into disjoint classes. The set of disjoint classes is denote  $S / \sim$ .

You can easily see that if two sequences converge to the same limit, they are necessarily in the same class.

**Definition:** A real number is an equivalence class in  $S / \sim$ .

So a real number should be thought of as the collection of all rational sequences which converge to it.

## Sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$

I skipped this slide in our class

Most of our definitions for sequences in  $\mathbb{R}$  are actually valid for sequences in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . Indeed, the only thing we really need to define the limit is the notion of distance. Thus if we replace the modulus function  $||$  on  $\mathbb{R}$  by the distance functions in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  all the definitions of convergent sequences and Cauchy sequences remain the same.

For instance, a sequence  $a(n) = (a(n)_1, a(n)_2)$  in  $\mathbb{R}^2$  is said to converge to a point  $l = (l_1, l_2)$  (in  $\mathbb{R}^2$ ) if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\sqrt{(a(n)_1 - l_1)^2 + (a(n)_2 - l_2)^2} < \epsilon$$

whenever  $n > N$ . A similar definition can be made in  $\mathbb{R}^3$  using the distance function on  $\mathbb{R}^3$ .

Theorems 2 (the Sandwich Theorem) and 3 (about monotonic sequences) don't really make sense for  $\mathbb{R}^2$  or  $\mathbb{R}^3$  because there is no ordering on these sets, that is, it doesn't really make sense to ask if one point on the plane or in space is less than the other.

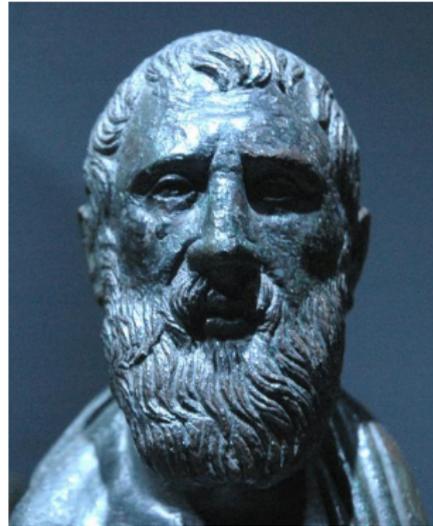
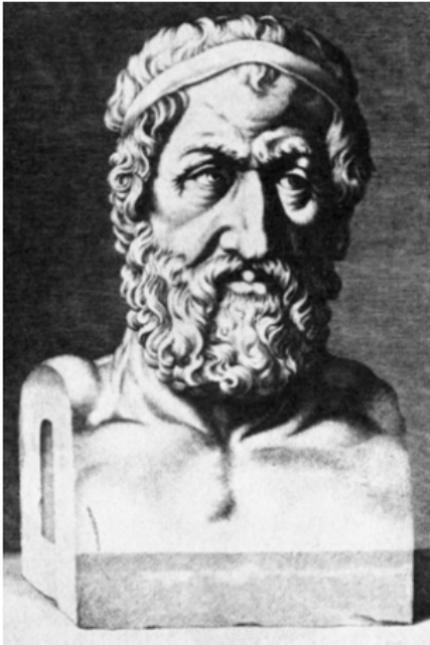
## The first man to think about limits?



Zeno of Elea (490 - 460 BCE) was a famous Greek philosopher (source: Wikipedia)

## Zeno of Elea

First let us record that we have no idea what Zeno looked like. The picture above was painted in the period 1588 - 1594 CE in Spain, about two thousand years after Zeno's time. Here are two more images of Zeno (also from Wikipedia)



## Zeno's Paradoxes

I couldn't find out where the first statue came from and when it was made. The second seems to have come from Herculaneum in Italy (incidentally, Elea (modern Vilia) is a town in Italy). Now Herculaneum was destroyed by a volcanic eruption from the nearby volcano Vesuvius in 79 CE, so it looks like the bust was created within 500 years of Zeno's death. Maybe it was even made during his lifetime and was lying around in some wealthy Roman's house for the next few centuries. Unfortunately, it is not clear whether this statue is one of Zeno of Elea or of another Zeno (of Citium) who lived about 150 years later. So we still really have no clue how he looked.

The important about Zeno is that it would appear that he was the first human to think about limits and limiting processes, at least in recorded history. Most of what we know about him is through his paradoxes, nine of which survive in the works of another famous Greek philosopher Aristotle (384 - 322 CE) , the official guru/tutor of Alexander the Great (aka Sikander in India).

## Achilles and the tortoise

One of Zeno's motivations for stating his paradoxes seems to have been to defend his own guru Parmenides' philosophy (whatever that was). Anyway here is his most famous paradox as recorded by Aristotle.

### Achilles and the tortoise:

In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.

Aristotle, Physics VI:9, 239b15

General knowledge question: Who was Achilles?

## Zeno's paradox animated

Achilles and the Tortoise