MA 105 D3 Lecture 4

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Formulæ for limits

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$$\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$$

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So if we are willing to skip a certain number of initial terms of a_n/b_n, the quotient will make perfect sense and the limit will be the quotient of the limits.

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Remark 3: Remember that when we defined sequences we defined them to be functions from $\mathbb{N}$ to X, for any set X. So far we have only considered $X = \mathbb{R}$, but as we said earlier we can take other sets, for instance, susbets of $\mathbb{R}$. For instance, if we take $X = \mathbb{R} \setminus 0$, Theorem 4 is not valid. The sequence $1/n$ is a Cauchy sequence in this X but obviously does not converge in X. If we take $X = \mathbb{Q}$, the example given in 1.5.(i) $(a_{n+1} = (a_n + 2/a_n)/2)$ is a Cauchy sequence in $\mathbb{Q}$ which does not converge in $\mathbb{Q}$. Thus Theorem 4 is really a theorem about real numbers.

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So a real number should be thought of as the collection of all rational sequences which converge to it.

Sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$

Most of our definitions for sequences in $\mathbb{R}$ are actually valid for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$. Indeed, the only thing we really need to define the limit is the notion of distance. Thus if we replace the modulus function $|\ |$ on $\mathbb{R}$ by the distance functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ all the definitions of convergent sequences and Cauchy sequences remain the same.

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For instance, a sequence $a(n) = (a(n)_1, a(n)_2)$ in $\mathbb{R}^2$ is said to converge to a point $l = (l_1, l_2)$ (in $\mathbb{R}^2$) if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sqrt{(a(n)_1 - l_1)^2 + a(n)_2 - l_2)^2} < \epsilon$$

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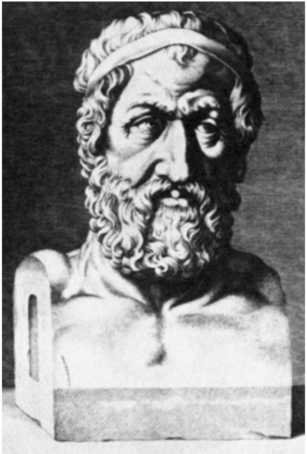
Theorems 2 (the Sandwich Theorem) and 3 (about monotonic sequences) don't really make sense for $\mathbb{R}^2$ or $\mathbb{R}^3$ because there is no ordering on these sets, that is, it doesn't really make sense to ask if one point on the plane or in space is less than the other.

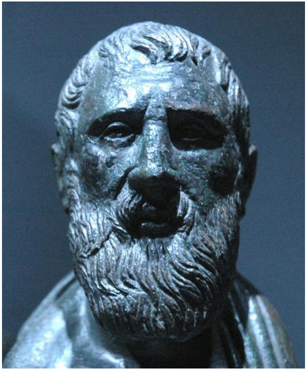
The first man to think about limits?

Zeno of Elea (490 - 460 BCE) was a famous Greek philosopher (source: Wikipedia)

Zeno of Elea

First let us record that we have no idea what Zeno looked like. The picture above was painted in the period 1588 - 1594 CE in Spain, about two thousand years after Zeno's time. Here are two more images of Zeno (also from Wikipedia)





Zeno's Paradoxes

I couldn't find out where the first statue came from and when it was made. The second seems to have come from Herculaneum in Italy (incidentally, Elea (modern Vilia) is a town in Italy). Now Herculaneum was destroyed by a volcanic erruption from the nearby volcano Vesuvius in 79 CE, so it looks like the bust was created within 500 years of Zeno's death. Maybe it was even made during his lifetime and was lying around in some wealthy Roman's house for the next few centuries. Unfortunately, it is not clear whether this statue is one of Zeno of Elea or of another Zeno (of Citium) who lived about 150 years later. So we still really have no clue how he looked.

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The important about Zeno is that it would appear that he was the first human to think about limits and limiting processes, at least in recorded history. Most of what we know about him is through his paradoxes, nine of which survive in the works of another famous Greek philosopher Aristotle (384 - 322 CE) , the official guru/tutor of Alexander the Great (aka Sikander in India).

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General knowledge question: Who was Achilles?

A gateway to infinite series

Nowadays, this line of argument does not really bother us, since we understand that an infinite number of terms (in this case consisting of the time travelled in each segment or the distance travelled in each segment) can add up to something finite. Nevertheless there are other philosophical issues that continued to bother mathematicians and physicists for a long time. After all, this kind of discussion does lead us to question whether intervals of time and space can be infinitely subdivided, or if "instantaneous motion" makes sense.

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Since we are learning mathematics, we won't speculate on physics or philosophy, but we note that Zeno's argument gives a good way to derive the sum of an infinite geometric series. The geometric series is one of the simplest examples of infinite series, so let us see how this is done.

Geometric series - the formula

Let us suppose that the speed of achilles is v and that the speed of the tortoise is rv for some $0 < r < 1$. We will assume that the tortoise was given a headstart of distance “a”.

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- Achilles catches up with the tortoise when $vt = a + rvt$, that is, at time $t = a/(v - rv)$ and when the total distance covered by Achilles is $vt = a/(1 - r)$.

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- Thus we get $a + ar + ar^2 + \cdots = a/(1 - r)$.

Infinite series - a more rigourous treatment

Let us recall what we mean when we write

$$a + ar + ar^2 + \ldots = \frac{a}{1-r}.$$

Another way of writing the same expression is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

The precise meaning is the following. Form the partial sums

$$s_n = \sum_{k=0}^{n} ar^k.$$

These partial sums $s_1, s_2, \ldots s_n, \ldots$ form a sequence and by $\sum_{k=0}^{\infty} ar^k = a/(1-r)$, we mean $\lim_{n\to\infty} s_n = a/1-r$.
So when we speak of the sum of an infinite series, what we really mean is the limit of its partial sums.

Convergence of the geometric series

So to justify our formula we should show that $\lim_{n\to\infty} s_n = a/1-r$, that is, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon,$$

for all $n > N$.

In other words we need to show that

$$\left| \frac{a(1-r^{n+1})}{1-r} - \frac{a}{1-r} \right| = \left| \frac{ar^{n+1}}{1-r} \right| < \epsilon$$

if n is chosen large enough.

But $\lim_{n\to\infty} r^n = 0$, so there exists N such that $r^{n+1} < (1-r)\epsilon/a$ for all $n > N$, so for this N, if $n > N$,

$$\left| s_n - \frac{a}{1-r} \right| < \epsilon.$$

This shows that the geometric series converges to the given expression.

The completeness of other spaces

Theorem 4, however, makes perfect sense - one can define Cauchy in $\mathbb{R}^2$ and $\mathbb{R}^3$ exactly as before, using the distance functions - and indeed, remains valid in $\mathbb{R}^2$ and $\mathbb{R}^3$.

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Finally, to emphasise that only the notion of distance matters for such questions we can define a distance function on $X = \mathcal{C}([a, b])$, the set of continuous functions from $[a, b]$ to $\mathbb{R}$, as follows:

$$\mathrm{dist}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

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$$\mathrm{dist}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Then Cauchy and convergent sequences in X can be defined as before, and we can prove (maybe next semester) that X is complete.

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for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.

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$$\lim_{x \to x_0} f(x) = l,$$

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or $f(x) \to l$ as $x \to x_0$ which we read as "$f(x)$" tends to l as x tends to x_0". This is just the rigourous way of saying that the

distance between $f(x)$ and l can be made as small as one pleases by making the distance between x and x_0 sufficiently small.

A subtle point and the rules for limits

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The rules and formulæ for limits of functions are the same as those for sequence and can be proved in almost exactly the same way. If $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$, then

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As before, implicit in the formulæ is the fact that if the limits on the left hand side exist. We prove the first rule below.

The proof of the addition formula for limits

Proof: We first show that $\lim_{x \to x_0} f(x) + g(x) = l_1 + l_2$. Let $\epsilon > 0$ be arbitrary.

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Since $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$, there exist δ_1, δ_2 such that

$$|f(x) - l_1| < \frac{\epsilon}{2} \quad \text{and} \quad |g(x) - l_2| < \frac{\epsilon}{2}$$

whenever $0 < |x - x_0| < \delta_1$ and $0 < |x - x_0| < \delta_2$.

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$$\begin{aligned} |f(x) + g(x) - (l_1 + l_2)| &= |f(x) - l_1 + g(x) - l_2| \\ &\leq |f(x) - l_1| + |g(x) - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which is what we needed to prove. If we replace $g(x)$ by $-g(x)$ we get the second part of the first rule. $\square$

The Sandwich Theorem(s) for limits of functions

Theorem 5: As $x \to x_0$, if $f(x) \to l_1$, $g(x) \to l_2$ and $h(x) \to l_3$ for functions f, g, h on some interval (a, b) such that $f(x) \le g(x) \le h(x)$ for all $x \in (a, b)$, then

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As before, we have a second version.

Theorem 6: Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l$ and If $g(x)$ is a function satisfying $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then $g(x)$ converges to a limit as $x \to x_0$ and

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Once again, note that we do not assume that $g(x)$ converges to a limit in this version of the theorem - we get the convergence of $g(x)$ for free .