MA 105 D3 Lecture 5

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Recap

Odds and ends about limits of functions of a real variable

Continuity

The rigourous definition of a limit of a function of a real variable

Definition: A function $f : (a, b) \to \mathbb{R}$ is said to tend to (or converge to) a limit l at a point $x_0 \in [a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

for all $x \in (a, b)$ such that $0 < |x - x_0| < \delta$.In this case, we write

$$\lim_{x \to x_0} f(x) = l,$$

or $f(x) \to l$ as $x \to x_0$ which we read as "$f(x)$" tends to l as x tends to x_0". Notice that in the definition above, the point x_0 can

be one of the end points a or b.

Thus the limit of a function may exist even if the function is not defined at that point.

Rules for limits, Sandwich theorems

1. $\lim_{x \to x_0} f(x) \pm g(x) = l_1 \pm l_2$.
2. $\lim_{x \to x_0} f(x)g(x) = l_1 l_2$.
3. $\lim_{x \to x_0} f(x)/g(x) = l_1/l_2$. provided $l_2 \neq 0$

Theorem 5: As $x \to x_0$, if $f(x) \to l_1$, $g(x) \to l_2$ and $h(x) \to l_3$ for functions f, g, h on some interval (a, b) such that $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then

$$l_1 \leq l_2 \leq l_3.$$

Theorem 6: Suppose $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l$ and If $g(x)$ is a function satisfying $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$, then $g(x)$ converges to a limit as $x \to x_0$ and

$$\lim_{x \to x_0} g(x) = l$$

Some examples

Let us look at Exercise 1.11. We will use this exercise to explore a few subtle points.

Let $c \in [a, b]$ and $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = 0$. Prove or disprove the following statements.

(i) $\lim_{x\to c}[f(x)g(x)] = 0$.

(ii) $\lim_{x\to c}[f(x)g(x)] = 0$, if g is bounded.($g(x)$ is said to be bounded on (a, b) if there exists $M > 0$ such that $|g(x)| < M$ for all $x \in (a, b)$.

(iii) $\lim_{x\to c}[f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists.

Before getting into proofs, let us guess whether the statements above are true or false.

(i) false

(ii) true

(iii) true.

(i) Notice that $g(x)$ is not given to be bounded - if this was not obvious before, you should suspect that such a condition is needed after looking at part (ii). So the most natural thing to do is to look for a counter-example to this statement by taking $g(x)$ to be an unbounded function. What is the simplest example of an unbounded function $g(x)$ on an open interval?

How about $g(x) = \frac{1}{x}$ on $(0, 1)$?
What would a candidate for $f(x)$ be - what is the simplest example of a function $f(x)$ which tends to 0 for some value of c in $[0, 1]$.

$f(x) = x$, and $c = 0$ is a pretty simple candidate.

Clearly $\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} 1 = 1 \neq 0$, which shows that (i) is not true in general.

Exercise 1: Can you find a counter-example to (i) with c in (a, b) (that is, c should not be one of the end points)? (Hint: Can you find an unbounded function on a closed interval $[a, b]$?)

Let us move to part (ii).
Suppose $g(x)$ is bounded on (a, b). This means that there is some real number $M > 0$ such that $|g(x)| < M$. Let $\epsilon > 0$. We would like to show that

$$|f(x)g(x) - 0| = |f(x)g(x)| < \epsilon,$$

if $0 < |x - c| < \delta$ for some $\delta > 0$.
Since $\lim_{x \to c} f(x) = 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon/M$ for all $0 < |x - c| < \delta$. It follows that

$$|f(x)g(x)| = |f(x)||g(x)| < \frac{\epsilon}{M} \cdot M = \epsilon$$

for all $0 < |x - c| < \delta$, and this is what we had to show.

Part (iii) follows immediately from the product rule, but can one deduce part (iii) from (ii) instead?
Hint: Think back to the lemma on convergent sequences that we proved in Lecture 1: Every convergent sequence is bounded. What is the analogue for functions which converge to a limit at some point? Indeed, you can easily show the following

Lemma 7: Let $f : (a, b) \to \mathbb{R}$ be a function such that $\lim_{x\to c} f(x)$ exists for some $c \in [a, b]$. If $c \in (a, b)$, there exists an (open) interval $I = (c - \eta, c + \eta) \subset (a, b)$ such that $f(x)$ is bounded on I. If $c = a$, then there is an open interval $I_1 = (a, a + \eta)$ such that $f(x)$ is bounded on I_1. Similarly if $c = b$, there exists an open interval $I_2 = (b - \eta, b)$ such that $f(x)$ is bounded on I_2.

The proof of the lemma above is almost the same as the the lemma for convergent sequences. Basically, replace "N" by "δ" in the proof.

If one applies the Lemma above to $g(x)$, we see that $g(x)$ is bounded in some (possibly) smaller interval $(0, \eta)$. Now apply part (ii) to this interval to deduce that (iii) is true.

Limits at infinity

There is one further case of limits that we need to consider. This occurs when we consider functions defined on open intervals of the form $(-\infty, b)$, (a, ∞) or $(-\infty, \infty) = \mathbb{R}$ and we wish to define limits as the variable goes to plus or minus infinity. The definition here is very similar to the definition we gave for sequences. Let us consider the last case.

Definition: We say that $f : \mathbb{R} \to \mathbb{R}$ tends to a limit l as $x \to \infty$ (resp. $x \to -\infty$) if for all $\epsilon > 0$ there exists $X \in \mathbb{R}$ such that

$$|f(x) - l| < \epsilon,$$

whenever $x > X$ (resp. $x < X$), and we write

$$\lim_{x \to \infty} f(x) = l \quad \text{or} \quad \lim_{x \to -\infty} f(x) = l.$$

or, alternatively, $f(x) \to l$ as $x \to \infty$ or as $x \to -\infty$, depending on which case we are considering.

Limits from the left and right

If $f:(a,b)\to\mathbb{R}$ is a function and $c\in(a,b)$, then it is possible to approach c from either the left or the right on the real line.

We can define the limit of the function $f(x)$ as x approaches c from the left (if it exists) as a number l^- such that for all $\epsilon>0$ there exists $\delta>0$ such that $|f(x)-l^-|<\epsilon$ whenever $|x-c|<\delta$ and $x\in(a,c)$.

Our notation for this is $\lim_{x\to c-} f(x)=l^-$, and it is also called the left hand (side) limit.

Exercise 2: Write down a definition for the limit of a function from the right. We usually denote the right hand (side) limit by $lim_{x\to c+} f(x)$. Show, using the definitions, that $\lim_{x\to c} f(x)$ exists if and only if the left hand and right hand limits both exist and are equal.

We can also think of the left hand limit as follows. We restrict our attention to the interval (a, c), that is we think of f as a function only on this interval. Call this restricted function f_a. Then, another way of defining the left hand limit is

$$\lim_{x \to c-} f(x) = \lim_{x \to c} f_a(x).$$

It should be easy to see that it is the same as the definition before One can make a similar definition for the right hand limit.

The notions of left and right hand limits are useful because sometimes a function is defined in different ways to the left and right of a particular point. For instance, $|x|$ has different definitions to the left and right of 0.

Calculating limits explicitly

As with sequences, using the rules for limits of functions together with the Sandwich theorem allows one to treat the limits of a large number of expressions once one knows a few basic ones:

(i) $\lim_{x\to 0} x^\alpha = 0$ if $\alpha > 0$, (ii) $\lim_{x\to\infty} x^\alpha = 0$ if $\alpha < 0$,
(iii) $\lim_{x\to 0} \sin x = 0$, (iv) $\lim_{x\to 0} \sin x/x = 1$
(v) $\lim_{x\to 0}(e^x - 1)/x = 1$, (vi) $\lim_{x\to 0} \ln(1+x)/x = 1$

We have not concentrated on trying to find limits of complicated expressions of functions using clever algebraic manipulations or other techniques. However, I can't resist mentioning the following problem.

Exercise 3: Find

$$\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}.$$

I will give the solution next time, together with the history of the problem (if I mention the history right away you will be able to get the solution by googling!), but feel free to use any method you like.

Continuity - the definition

Definition: If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Thus, if c is one of the end points we require only the left or right hand limit to exist.

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

If f is not continuous at a point c we say that it is discontinuous at c, or that c is a point of discontinuity for f.

Intuitively, continuous functions are functions whose graphs can be drawn on a sheet of paper without lifting the pencil of the sheet of paper. That is, there should be no “jumps” in the graph of the function.

Continuity of familiar functions: polynomials

What are the functions we really know or understand? What does "knowing" or understanding a function $f(x)$ even mean? Presumably, if we understand a function f, we should be able to calculate the value of the function $f(x)$ at any given point x. But if you think about it, for what functions $f(x)$ can you really do this?

One class of functions is the polynomial functions. More generally we can understand rational functions, that is functions of the form $R(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, since we can certainly compute the values of $R(x)$ by plugging in the value of x. How do we show that polynomials or rational functions are continuous (on $\mathbb{R}$)?

It is trivial to show from the definition that the constant functions and the function $f(x) = x$ are continuous. Because of the rules for limits of functions, the sum, difference, product and quotient (with non-zero denominator) of continuous functions are continuous. Applying this fact we see easily that $R(x)$ is continuous whenever the denominator is non-zero.

Continuity of other familiar functions

What are the other (continuous) functions we know? How about the trignometric functions? Well, here it is less clear how to proceed. After all we can only calculate $\sin x$ for a few special values of x ($x = 0, \pi/6, \pi/4, \ldots$ etc.). How can we show continuity when we don't even know how to compute the function?

Of course, if we define $\sin x$ as the y-coordinate of a point on the unit circle it seems intuitively clear that the y-coordinate varies continuously as the point varies on the unit circle, but knowing the precise definition of continuity this argument should not satisfy you.

We will not prove the continuity of $\sin x$ in this course, though we will given an idea of how this is done next week. So let us assume from now on that $\sin x$ is continuous. How can we show that $\cos x$ is continuous?