MA 105 D3 Lecture 6

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Continuity - the definition

Definition: If $f : [a, b] \to \mathbb{R}$ is a function and $c \in [a, b]$, then f is said to be continuous at the point c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Thus, if c is one of the end points we require only the left or right hand limit to exist.

A function f on (a, b) (resp. $[a, b]$) is said to be continuous if and only if it is continuous at every point c in (a, b) (resp. $[a, b]$).

If f is not continuous at a point c we say that it is discontinuous at c, or that c is a point of discontinuity for f.

Intuitively, continuous functions are functions whose graphs can be drawn on a sheet of paper without lifting the pencil of the sheet of paper. That is, there should be no "jumps" in the graph of the function.

The composition of continuous functions

Theorem 8: Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be functions such that f is continuous at x_0 in (a, b) and g is continuous at $f(x_0) = y_0$ in (c, d). Then the function $g(f(x))$ (also written as $g \circ f(x)$ sometimes) is continuous at x_0. So the composition of continuous functions is continuous.

Exercise 4: Prove the theorem above starting from the definition of continuity.

Using the theorem above we can show that $\cos x$ is continuous if we show that $\sqrt{x}$ is continuous, since $\cos x = \sqrt{1 - \sin^2 x}$ and we know that $1 - \sin^2 x$ is continuous since it is the product of the sums of two continuous functions ($(1 + \sin x)$ and $(1 - \sin x)$!).

Once we have the continuity of $\cos x$ we get the continuity of all the rational trignometric functions, that is functions of the form $P(x)/Q(x)$, where P and Q are polynomials in $\sin x$ and $\cos x$, provided $Q(x)$ is not zero.

The continuity of the square root function

Thus in order to prove the continuity of $\cos x$ (assuming the continuity of $\sin x$) we need only prove the continuity of the square root function.

The main observation is that continuity is a local property, that is, only the behaviour of the function near the point being investigated is important.

Let $x_0 \in [0, \infty)$. To show that the square root function is continuous at x_0 we need to show that $\lim_{y \to x_0} \sqrt{y} = \sqrt{x_0}$, that is we need to show that $|\sqrt{y} - \sqrt{x_0}| < \epsilon$ whenever $0 < |y - x_0| < \delta$ for some δ. First assume that $x_0 \neq 0$. Then

$$|\sqrt{y} - \sqrt{x_0}| = \left| \frac{y - x_0}{\sqrt{y} + \sqrt{x_0}} \right| < \frac{|y - x_0|}{\sqrt{x_0}}.$$

If we choose $\delta = \epsilon\sqrt{x_0}$, we see that

$$|\sqrt{y} - \sqrt{x_0}| < \epsilon,$$

which is what we needed to prove. When $x_0 = 0$, I leave the proof as an exercise. $\square$

The intermediate value theorem

One of the most important properties of continuous functions is the Intermediate Value Property (IVP). We will use this property repeatedly to prove other results.

Theorem 9: Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. For every u between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ there such that $f(c) = u$.

Functions which have this property are said to have the Intermediate Value Property. Theorem 9 can thus be restated as saying that continuous functions have the IVP.

We will not be proving this property - it is a consequence of the completeness of the real numbers. Intuitively, this is clear. Since one can draw the graph of the function without lifting one's pencil off the sheet of paper, the pencil must cut every line $y = e$ with e between $f(a)$ and $f(b)$.

The IVT in a picture

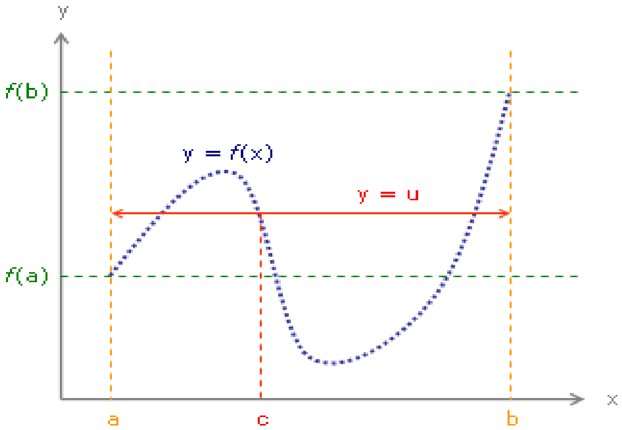

Image created by Enoch Lau see
http://en.wikipedia.org/wiki/File:Intermediatevaluetheorem.png
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Zeros of functions

One of the most useful applications of the intermediate value property is to find roots of polynomials, or, more generally, to find zeros of continuous functions, that is to find points $x \in \mathbb{R}$ such that $f(x) = 0$.

Theorem 10: Every polynomial of odd degree has at least one real root.

Proof: Let $P(x) = a_n x^n + \ldots + a_0$ be a polynomial of odd degree. We can assume without loss of generality that $a_n > 0$. It is easy to see that if we take $x = b > 0$ large enough, $P(b)$ will be positive. On the other hand, by taking $x = a < 0$ small enough, we can ensure that $P(a) < 0$. Since $P(x)$ is continuous, it has the IVP, so there must be a point $x_0 \in (a, b)$ such that $f(x_0) = 0$. $\square$

The IVP can often be used to get more specific information. For instance, it is not hard to see that the polynomial $x^4 - 2x^3 + x^2 + x - 3$ has a root that lies between 1 and 2.

Continuous functions on closed, bounded intervals

The other major result on continuous functions that we need is the following. A closed bounded interval is one of the form $[a, b]$, where $-\infty < a$ and $b < \infty$.

Theorem 11: A continuous function on a closed bounded interval $[a, b]$ is bounded and attains its infimum and supremum, that is, there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, where m and M denote the infimum and supremum respectively.

Again, we will not prove this, but will use it quite often. Note the contrast with open intervals. The function $1/x$ on $(0, 1)$ does not attain a maximum - in fact it is unbounded. Similarly the function $1/x$ on $(1, \infty)$ does not attain its minimum, although, it is bounded below.

Exercise 5: In light of the above theorem, can you find a continuous function $g : (a, b) \to \mathbb{R}$ for part (i) of Exercise 1.11, with $c \in (a, b)$?

The function $\sin \frac{1}{x}$

Let us look at Exercise 1.13 part (i).
Consider the function defined as $f(x) = \sin \frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$. The question asks if this function is continuous at $x = 0$.
How about $x \neq 0$? Why is $f(x)$ continuous? Because it is a compostion of the sin function and a rational function $1/x$. Since both of these are continuous when $x \neq 0$, so is $f(x)$.

Let us look at the sequence of points $x_n = 2/(2n+1)\pi$. Clearly $x_n \to 0$ as $n \to \infty$.
For these points $f(x_n) = \pm 1$. This means that no matter how small I take my δ, there will be a point $x_n \in (0, \delta)$, such that $|f(x_n)| = 1$.
But this means that $|f(x) - f(0)| = |f(x)|$ cannot be made smaller than 1 no matter how small δ may be. Hence, f is not continuous at 0. The same kind of argument will show that there is no value that we can assign $f(0)$ to make the function $f(x)$ continuous at 0.

You can easily check that $f(x)$ has the IVP. However, we have proved that it is not continuous. So IVP $\not\Rightarrow$ continuity.

Sequential continuity

The preceeding example showed that in order to demonstrate that a function, say $f(x)$, is not continuous at a point x_0 it is enough to find a sequence x_n tending to x_0 such that the value of the function $|f(x_n) - f(x_0)|$ remains large. Suppose it is not possible to find such a sequence. Does that mean the function is continuous at x_0? The following theorem answers the question affirmatively.

Theorem 12: A function $f(x)$ is continuous at a point a if and only if for every sequence $x_n \to a$, $\lim_{x_n \to a} f(x_n) = f(a)$.

A function that satisfies the property that for every sequence $x_n \to a$, $\lim_{x_n \to a} f(x_n) = f(a)$ is said to be sequentially continuous. The theorem says that sequential continuity and continuity are the same thing. Indeed, it is clear that a continuous function is necessarily sequentially continuous. It is the reverse that is slightly harder to prove.

Limits of functions of several variables

Just like we did for sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$, we can define the notion of the limit of a function for functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The function $f(x_1, x_2)$ is said to tend to a limit l as $(x_1, x_2) \to (a_1, a_2)$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x_1, x_2) - l| < \epsilon$$

whenever $0 < \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta$. Notice, that one can now approach the point (a_1, a_2) from any direction in the plane. Our definition requires that the limits from the different directions all exist and be equal. This is quite a powerful condition.

If we have functions from $\mathbb{R}^2 \to \mathbb{R}^2$ we can make exactly the same definition. But this time $l = (l_1, l_2)$ will be in $\mathbb{R}^2$ and so will $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, so we will have to replace the modulus function by the distance between these two quantities:

$$\sqrt{[f_1(x_1, x_2) - l_1]^2 + [f_2(x_1, x_2) - l_2]^2}.$$

Limits of functions of several variables

The definitions we have made go through for functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, where m and n may be different. For instance, we have considered the case when $m = 2$ and $n = 1$ and also the case $m = 2$ and $n = 2$ above. But we could allow m and n to take any of the values 1, 2 or 3 (in fact, we can allow values greater than 3 as well!).

Exercise 1: Show that

$$\lim_{y \to x} f(y) = l$$

for $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2 \iff$

$$\lim_{y \to x} f_1(y) = l_1 \quad \text{and} \quad \lim_{y \to x} f_2(y) = l_2,$$

where $l = (l_1, l_2)$ In other words, when dealing with limits of functions which are vector-valued, it is enough to study the limits of the coordinate functions.

Continuous functions of several variables

Once the definition of the limit is clear it makes sense to talk of continuity as well. All the definitions remain the same, only the definition of the distance function changes depending on the domain and the range.

For instance, provided we know what "closed and bounded sets" are in $\mathbb{R}^2$ or $\mathbb{R}^3$, Theorem 11 goes through for continuous functions $f : \mathbb{R}^m \to \mathbb{R}$. $(m = 2, 3)$. For functions with more than one variable in the range the first part of Theorem 11 still works, but for the second part things are more compacted (again there is no "ordering" in $\mathbb{R}^2$ or $\mathbb{R}^3$).

While it is easy to see what a bounded set in $\mathbb{R}^m$ should be, closed is a little more complicated and we will not give the definition here. However, a rectangle of the form $[a, b] \times [c, d]$ in $\mathbb{R}^2$ is an example of a closed and bounded set (also called "compact sets" of this form).
Theorem 12 goes through without any problems even when the range is in $\mathbb{R}^2$ or $\mathbb{R}^3$.