# MA 105 D3 Lecture 8

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Recap

Properties of the first derivative

Maxima/Minima - beyond the first derivative

Concavity and convexity

Set Cardinality

# The theorems of Fermat and Lagrange

Theorem 13: If $f : X \to \mathbb{R}$ is differentiable and has a local minimum or maximum at a point $x_0 \in X$, $f'(x_0) = 0$.

Note that for $x_0$ to be a local minimum/maximum there must be an open interval containing $x_0$ such that $f(x_0) \leq / \geq f(x)$ for all $x$ in that interval. So if $X$ is a closed interval $[a, b]$, Fermat's theorem says nothing about the end points $a$ and $b$.

Theorem 15: Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and that $f$ is differentiable in $(a, b)$. Then there is a point $x_0$ in $(a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0).$$

# Not so well-behaved functions

The standard example of a function that is continuous but not differentiable at some point is the function $|x|$ in some interval containing 0.

At 0 both the left and right hand derivatives exist, but they are not equal.

Thus, roughly speaking, non-differentiable functions are those whose graphs have sharp corners.

Exercise: (hard!) Can you find a continuous function on some interval which is everywhere continuous but nowhere differentiable?

# The sign of the derivative

Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at some point $x_0$ in $(a, b)$ and suppose that $f'(x_0) > 0$. Consider (as we have done in the proof of Fermat's theorem) the quotient

$$\frac{f(x_0 + h) - f(x_0)}{h}.$$

We know that for $|h|$ small enough, say $|h| < \delta$ for some $\delta$, the quotient above must be positive (why?). Thus if $-\delta < h < 0$, we must have $f(x_0 + h) - f(x_0) < 0$. Similarly, if $0 < h < \delta$, we see that $f(x_0 + h) - f(x_0) > 0$.

This shows that $f(x)$ is increasing at $x_0$. Similarly if $f'(x_0) < 0$ is decreasing at $x_0$.

# Darboux's Theorem

Another interesting property of differentiable functions is that their derivatives have the IVP (intermediate value property). This fact is sometimes called Darboux's Theorem.

Theorem 17: Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If $c, d$, $c < d$ are points in $(a, b)$, then for every $u$ between $f'(c)$ and $f'(d)$, there exists an $x$ in $[c, d]$ such that $f'(x) = u$.

Proof: We can assume, without loss of generality, that $f'(c) < u < f'(d)$, otherwise we can take $x = c$ or $x = d$. Define $g(t) = ut - f(t)$. This is a continuous function on $[c, d]$, and hence, by Theorem 11, must attain its extreme values. These extreme values cannot occur at $c$ or $d$ since $g'(c) = u - f'(c) > 0$ and $g'(d) = u - f'(d) < 0$ (contradicts Fermat's Theorem). It follows that there exists $x \in (c, d)$ where $g$ takes an extreme value. By Fermat's Theorem $g'(x) = 0$ which yields $f'(x) = u$. $\square$

# Continuity of the first derivative

We have just seen that the derivative satisfies the IVP. Can we find a function which is differentiable but for which the derivative is not continuous?

Here is the standard example:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0. \end{cases}$$

This function will be differentiable at 0 but its derivative will not be continuous at that point. In order to see this you will need to study the function in Exercise 1.13(ii). This will show that $f'(0) = 0$. On the other hand, if we use the product rule when $x \neq 0$ we get

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

which does not go to 0 as $x \to 0$.

# Back to maxima and minima

We will assume that $f : [a, b] \to \mathbb{R}$ is a continuous function and that $f$ is differentiable on $(a, b)$. A point $x_0$ in $(a, b)$ such that $f'(x_0) = 0$ often called a stationary point. We will assume further that $f'(x)$ is differentiable at $x_0$, that is, that the second derivative $f''(x_0)$ exists. We formulate the Second Derivative Test below.

Theorem 18: With the assumptions above:

1. If $f''(x_0) > 0$, the function has a local minimum at $x_0$.
2. If $f''(x_0) < 0$, the function has a local maximum at $x_0$.
3. If $f''(x_0) = 0$, no conclusion can be drawn.

# The proof of the Second Derivative Test

Proof: The proofs are straightforward. For instance, to prove the first part we observe that

$$0 < f''(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h)}{h}.$$

It follows that for $|h|$ small enough, $f'(x_0 + h) < 0$, if $h < 0$ and $f'(x_0 + h) > 0$ if $h > 0$. It follows that $f(x_0)$ is decreasing to the left of $x_0$ and increasing to the right of $x_0$. Hence, $x_0$ must be a local minimum. A similar argument yields the second case. $\square$

If the third case of the theorem above occurs, the function may be changing from concave to convex. In this case $x_0$ is called a point of inflection. An example of this phenomenon is given by $f(x) = x^3$ at $x = 0$.

# Concavity and convexity

Let $I$ denote an interval (open or closed or half-open).
Definition: A function $f : I \to \mathbb{R}$ is said to be concave (or sometimes concave downwards) if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$$

for all $x_1$ and $x_2$ in $I$ and $t \in [0,1]$. Similarly, a function is said to be convex (or concave upwards) if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

By replacing the $\geq$ and $\leq$ signs above by strict inequalities we can define strictly concave and strictly convex functions.

For various reasons, convex functions are more important in mathematics than concave functions and for this reason we will concentrate on the former rather than the latter. On the other hand, note that if $f(x)$ is a concave function, $-f(x)$ is a convex function, so it is really enough to study one class or the other.

# Examples of concave and convex functions

Here are some examples of convex functions.

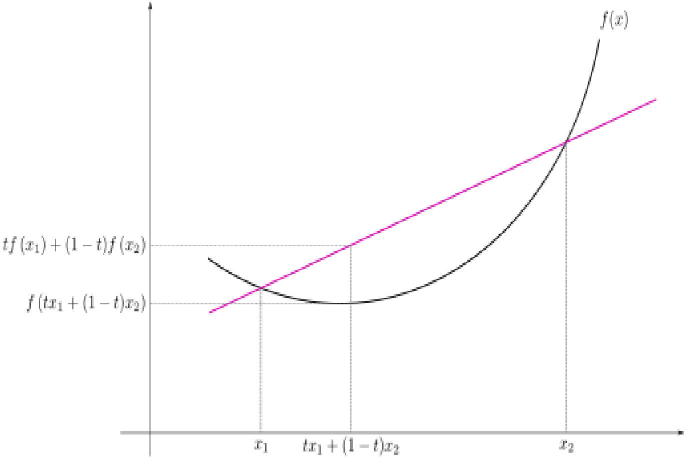
1. $f(x) = x^2$ on $\mathbb{R}$.
2. $f(x) = x^3$ on $[0, \infty)$.
3. $f(x) = e^x$ on $\mathbb{R}$.

Examples of concave functions include

1. $f(x) = -x^2$
2. $f(x) = x^3$ on $(-\infty, 0]$
3. $f(x) = \log x$ on $(0, \infty)$.

For a convex function $f$ and point $c \in (x_1, x_2)$, the point $(c, f(c))$ always lies below the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

# Convexity illustrated graphically



# Propeties of Convex functions

Convex functions have many nice properties. For instance, it is easy to show that convex functions are continuous (do this!). More is true.

Exercise 1. Every convex function is Lipschitz continuous (a function is Lipschitz continuous if it satisfies the inequality given in Exercise 1.16 but with $\alpha = 1$). In fact, much more is true. A convex function is actually differentiable at all but at most countably many points.

A differentiable function is convex if and only if its derivative is monotonically increasing. Moreover, if a function is both differentiable and convex, it is continuously differentiable, that is, its derivative is continuous (feel free to try proving these facts).

# Convexity and the second derivative

A twice differentiable function on an interval will be convex if its second derivative is everywhere non-negative. If the second derivative is positive, the function will be strictly convex.

However, the converse of the second statement above is not true. Can you give a counter-example to the converse of the second statement?

How about $f(x) = x^4$?

Definition: A point of inflection $x_0$ for a function $f$ is a point where the function changes its behavior from concave to convex (or vice-versa). At such a point, if $f$ is twice differentiable, $f''(x_0) = 0$, but this is only a necessary, not a sufficient condition.(Why?) If further, we also assume that the lowest order ($\geq 2$) non-zero derivative is odd, then we get a sufficient condition.

# Counting versus pairing

Having names for and dealing with large numbers (lakhs, millions, crores, billions etc.) is a very recent phenomenon. For most of human history, people did not have names for numbers beyond one hundred or a thousand. Indeed, supposedly some "primitive" tribes only had names for "one" and "two". Everything else was "many".

Nevertheless if you gave a stone-age man a herd of cattle and flock of sheep and asked him which had more animals, he could certainly have told you even if he didn't have names for the numbers of the cattle and the sheep. How?

He would simply have taken one cow from the herd and paired it with one sheep from the flock, and he would have continued to pair the animals in twos until either the cattle or the sheep ran out.

# What do we mean by a cardinal number?

If the cows and sheep are exactly paired our stone-age man can say that there are the same number of cows and sheep.

Note that he does not need to count the number of cows individually (or count the number of sheep). He does not need names for numbers at all!

So what do we mean by a cardinal number? For instance, what do we mean by “seven”. It is the name we give for a certain property of various collections of objects- cows, sheep, books, ships, colours, chemical compounds, grains of rice. The objects of any one collection can be paired off exactly with objects in one of the other collections.

# Cardinality: the definition

So "seven" is the property that all possible collections with a certain number of objects (namely, seven!) have. It is a completely abstract concept. "Sevenness" is the property that all of these sets of objects have.

Definition: Two sets $X$ and $Y$ are said to have the same cardinality if there is a bijective function from $X$ to $Y$.

All of this discussion probably feels obvious to you. The discussion gets more tricky (and thus more interesting!) when $X$ and $Y$ are infinite sets.

# Infinite sets

Example: Let $X = \mathbb{N}$ and let $Y$ be set of all perfect squares of numbers in $\mathbb{N}$. Clearly $X$ and $Y$ have the same cardinality since the function $f(n) = n^2$ gives a bijective map from $X$ to $Y$.

But $Y$ is a subset of $X$! So a set and a proper subset can have the same cardinality! This could never happen if $X$ and $Y$ were finite!

A set that has the same cardinality as the set $\mathbb{N}$ is called countable or denumerable. Thus the set of perfect squares is countable. How about the set of odd natural numbers?The set of integers?The set of rational numbers $\mathbb{Q}$?