Department of Mathematics
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MA 109: Calculus I

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Website

For course materials and updates please check `http://www.math.iitb.ac.in/~ravir/ma109index.html`
These materials and updates will also be posted on "moodle", an online interface for the course. This should be already functional.

Syllabus

- The convergence of sequences and series.
- A review of limits, continuity, differentiability.
- The Mean Value Theorem, Taylor's theorem, power series, maxima and minima.
- Riemann integrals, The Fundamental Theorem of Calculus, improper integrals; applications to area and volume.
- Partial derivatives, the gradient and directional derivatives, the chain rule, maxima and minima in several variables, Lagrange multipliers.

Texts/References

[Apo80] T.M. Apostol, Calculus, Volumes 1 and 2, 2nd ed., Wiley (2007).

[Ste03] James Stewart, Calculus, 8th ed., Thomson (2011).

[TF98] G.B. Thomas and R.L. Finney, Calculus and Analytic Geometry, 12th ed., Pearson (2015).

Policy on Attendance

Students are expected to attend all lectures and tutorial sessions. However, we do understand that many of you may face power cuts, unstable internet connections or other constraints. In that case, please make sure that you watch the recordings of the zoom lectures and tutorials. Please try to access the moodle site whenever your internet connections lets you do so.

Evaluation Plan

The evaluation plan has not been finalised since we are not sure of the quality of online access of our students. We are likely to have weekly short quizes of about 15 minutes each. There maybe also be one longer quiz of about one hour. The quizzes are likely to take place on the moodle learning platform (`https://moodle.iitb.ac.in/login/index.php`) or on another platform called SAFE. You will need to download the SAFE app onto your mobile phones.

The exact dates of the quizzes and exams will be announced in class and also sent to you via email. There will also be announcements via moodle.

The final exam is likely to account for about 40%-50% of the total marks for the course.
All of these are tentative plans. We are likely to have a clearer plan after the first week or two of classes.

Tutorial sheet 1: Sequences, limits, continuity, differentiability

Sequences

1. Using the (ϵ-N) definition of a limit, prove the following:

   (i) $\lim\limits_{n\to\infty} \dfrac{10}{n} = 0$

   (ii) $\lim\limits_{n\to\infty} \dfrac{5}{3n+1} = 0$

   (iii) $\lim\limits_{n\to\infty} \dfrac{n^{2/3}\sin(n!)}{n+1} = 0$

   (iv) $\lim\limits_{n\to\infty} \left(\dfrac{n}{n+1} - \dfrac{n+1}{n}\right) = 0$

2. Show that the following limits exist and find them:

   (i) $\lim\limits_{n\to\infty} \left(\dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \cdots + \dfrac{n}{n^2+n}\right)$

   (ii) $\lim\limits_{n\to\infty} \left(\dfrac{n!}{n^n}\right)$

   (iii) $\lim\limits_{n\to\infty} \left(\dfrac{n^3+3n^2+1}{n^4+8n^2+2}\right)$

   (iv) $\lim\limits_{n\to\infty} (n)^{1/n}$

   (v) $\lim\limits_{n\to\infty} \left(\dfrac{\cos \pi\sqrt{n}}{n^2}\right)$

   (vi) $\lim\limits_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)\right)$

3. Show that the following sequences are not convergent:

   (i) $\left\{\dfrac{n^2}{n+1}\right\}_{n\geq 1}$ (ii) $\left\{(-1)^n\left(\dfrac{1}{2} - \dfrac{1}{n}\right)\right\}_{n\geq 1}$

4. Determine whether the sequences are increasing or decreasing:

   (i) $\left\{\dfrac{n}{n^2+1}\right\}_{n\geq 1}$

   (ii) $\left\{\dfrac{2^n 3^n}{5^{n+1}}\right\}_{n\geq 1}$

   (iii) $\left\{\dfrac{1-n}{n^2}\right\}_{n\geq 2}$

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

   (i) $a_1 = \dfrac{3}{2}, a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right) \quad \forall\, n \geq 1$

   (ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n}\ \forall\ \ n \geq 1$

   (iii) $a_1 = 2,\ a_{n+1} = 3 + \dfrac{a_n}{2}\ \forall\, n \geq 1$

6. If $\lim_{n\to\infty} a_n = L$, find the following: $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty} |a_n|$.

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \ \forall\, n \geq n_0.$$

8. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$. State and prove a corresponding result if $a_n \to L > 0$.

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

   (i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.

   (ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent iff both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

Limits of functions of a real variable, continuity, differentiability

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a, b]$. Prove or disprove the following statements.

   (i) $\lim_{x\to c}[f(x)g(x)] = 0$.

   (ii) $\lim_{x\to c}[f(x)g(x)] = 0$, if g is bounded.

   (iii) $\lim_{x\to c}[f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for some $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0}[f(\alpha + h) - f(\alpha - h)] = 0.$$

   Analyze the converse.

13. Discuss the continuity of the following functions:

   (i) $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

   (ii) $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

   (iii) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if } \ 1 \leq x < 2, \\ 1 & \text{if } \ x = 2, \\ \sqrt{6-x} & \text{if } \ 2 < x \leq 3. \end{cases}$

14. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.

15. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is $f^{'}$ a continuous function?

16. Let $f : (a, b) \to \mathbb{R}$ be a function such that

$$|f(x+h) - f(x)| \leq C|h|^{\alpha}$$

for all $x, x+h \in (a, b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f^{'}(x)$ for $x \in (a, b)$.

17. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f^{'}(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]

18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f^{'}(c) = f^{'}(0)f(c)$.

19. Using the theorem on derivative of inverse function, compute the derivative of (i) $\cos^{-1} x, \; -1 < x < 1$. (ii) $\operatorname{cosec}^{-1} x, \; |x| > 1$.

20. Compute $\frac{dy}{dx}$, given

$$y = f\left(\frac{2x-1}{x+1}\right) \text{ and } f^{'}(x) = \sin(x^2).$$

Supplement

1. A sequence $\{a_n\}_{n \geq 1}$ is said to be Cauchy if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon, \; \forall \, m, n \geq n_0$. In other words, if we choose n_0 large enough, we can make sure that the elements of a Cauchy sequence are close to each other as we want beyond n_0. One can show that a sequence in $\mathbb{R}$ is convergent if and only if it is Cauchy. To show that a convergent sequence in $\mathbb{R}$ is Cauchy is easy. To show that every Cauchy sequence in $\mathbb{R}$ converges is harder, and moreover, involves making a precise definition of the set of real numbers. Sets in which every Cauchy sequence converges are called *complete*. Thus the set of real numbers is complete.

2. To prove that a sequence $\{a_n\}_{n \geq 1}$ is convergent to a limit L, one needs to first guess what this limit L might be and then verify the required property. However the concept of 'Cauchyness' of a sequence is an intrinsic property, that is, we can decide whether a sequence is Cauchy by examining the sequence itself. There is no need to guess what the limit might be.

3. In problem 5(i), we defined

$$a_1 = \frac{3}{2}, \;\; a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \;\; \forall \, n \geq 1.$$

The sequence $\{a_n\}_{n\geq 1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n\geq 1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find a rational approximation (in computing machines) of $\sqrt{2}$.

Optional Exercises:

1. Show that the function f in Question 14 satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.

2. Show that in Question 18, f has a derivative of every order on $\mathbb{R}$.

3. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere and is differentiable everywhere except at 2 points.

4. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$ Show that f is discontinuous at every $c \in \mathbb{R}$.

5. Let $g(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$ Show that g is continuous only at $c = 1/2$.

6. Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$ be such that $\lim_{x \to c} f(x) > \alpha$. Prove that there exists some $\delta > 0$ such that
$$f(c + h) > \alpha \text{ for all } 0 < |h| < \delta.$$

7. Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent:
   (i) f is differentiable at c.
   (ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and
   $$f(c + h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta).$$
   (iii) There exists $\alpha \in \mathbb{R}$ such that
   $$\lim_{h \to 0} \left(\frac{|f(c + h) - f(c) - \alpha h|}{|h|} \right) = 0.$$

8. Suppose f is a function that satisfies the equation $f(x + y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose also that
$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$
   Find $f(0), f'(0), f'(x)$.

9. Suppose f is a function with the property that $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. Show that $f(0) = 0$ and $f'(0) = 0$.

10. Show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

Tutorial sheet 2: Rolle's theorem, MVT, maxima/minima

1. Show that all the roots of the cubic $x^3 - 6x + 3$ are real.

2. Let p and q be two real numbers with $p > 0$. Show that the cubic $x^3 + px + q$ has exactly one real root.

3. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f^{'}(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:

   (i) $p < 0$.

   (ii) f has a local maximum/minimum at $\pm\sqrt{-p/3}$.

   (iii) The maximum/minimum values are of opposite signs.

5. Use the MVT to prove that $|\sin a - \sin b| \leq |a - b|$, for all $a, b \in \mathbb{R}$.

6. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f^{'}(c_1) + f^{'}(c_2) = 2$.

7. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f^{'}(x)$ exists and $f^{'}(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$. Is it true that $f(x) = x$ for every x?

8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.

   (i) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 1$

   (ii) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 2$

   (iii) $f^{''}(x) \geq 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 100$ for all $x > 0$

   (iv) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 1$ for all $x < 0$

9. Let $f(x) = 1+12|x|-3x^2$. Find the global maximum and the global minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.

10. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

    (i) $y = 2x^3 + 2x^2 - 2x - 1$

    (ii) $y = 1 + 12|x| - 3x^2, \ x \in [-2, 5]$

11. Sketch a continuous curve $y = f(x)$ having all the following properties:

    $f(-2) = 8, \ f(0) = 4, \ f(2) = 0; \ \ f^{'}(2) = f^{'}(-2) = 0;$

    $f^{'}(x) > 0$ for $|x| > 2, \ f^{'}(x) < 0$ for $|x| < 2;$

    $f^{''}(x) < 0$ for $x < 0$ and $f^{''}(x) > 0$ for $x > 0$.

12. Give an example of $f : (0, 1) \to \mathbb{R}$ such that f is

    (i) strictly increasing and convex.

(ii) strictly increasing and concave.

(iii) strictly decreasing and convex.

(iv) strictly decreasing and concave.

13. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?

    (i) If f and g have a local maximum at $x = c$, then so does h.

    (ii) If f and g have a point of inflection at $x = c$, then so does h.

14. Sketch the curve following the template of exercise 10: $y = \dfrac{x^2}{x^2+1}$

Tutorial sheet 3: Supplement on Taylor series

In this tutorial sheet, we will intersperse the exercises with the text, so you will have to read through the sheet somewhat carefully.

The Kerala School of Mathematics

In the fourteenth century CE, mathematicians in Kerala made a number of remarkable discoveries. Sangamagrāma Mādhavan (1350-1425 CE) appears to have been one of the founders of what is now known as the Kerala School of Mathematics, anticipating many of the later developments in Europe. The following is an extract from `http://en.wikipedia.org/wiki/Madhava_of_Sangamagrama`:

Among his many contributions, he discovered the infinite series for the trigonometric functions of sine, cosine, tangent and arctangent, and many methods for calculating the circumference of a circle. One of Madhava's series is known from the text Yuktibhāṣā, which contains the derivation and proof of the power series for inverse tangent, discovered by Madhava. In the text, Jyeṣṭhadeva describes the series in the following manner:
"The first term is the product of the given sine and radius of the desired arc divided by the cosine of the arc. The succeeding terms are obtained by a process of iteration when the first term is repeatedly multiplied by the square of the sine and divided by the square of the cosine. All the terms are then divided by the odd numbers $1, 3, 5, \ldots$. The arc is obtained by adding and subtracting respectively the terms of odd rank and those of even rank. It is laid down that the sine of the arc or that of its complement whichever is the smaller should be taken here as the given sine. Otherwise the terms obtained by this above iteration will not tend to the vanishing magnitude."

Exercise 1. *Write down the Taylor series for (i)* $\cos x$*, (ii)* $\arctan x$ *about the point* 0*. Write down a precise remainder term* $R_n(x)$ *in each case.*

Exercise 2. *Our examples of Taylor's series have usually been series about the point* 0*. Write down the Taylor series of the polynomial* $x^3 - 3x^2 + 3x - 1$ *about the point* 1*.*

Exercise 3. *What is the Taylor series of the function* $1729x^{1729} + 1728x^{1728} + 1000x^{1000} + 729x^{729} + 1$ *about the point* 0*?*

Power series

Exercise 4. *Consider the series* $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ *for a fixed* x*. Prove that it converges as follows. Choose* $N > 2x$*. We see that for all* $n > N$*,*

$$\frac{x^{n+1}}{(n+1)!} < \frac{1}{2} \cdot \frac{x^n}{n!}.$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$*), convergent.*

Taylor series (or more generally "power series") can be differentiated and integrated "term by term". That is if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

And similarly,

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} a_n \int_a^b x^n dx.$$

We will not be proving these facts but you can use them below.

Exercise 5. *Using Taylor series write down a series for the integral*

$$\int \frac{e^x}{x} dx.$$

Exercise 6. *Use Taylor series to approximate* $\int_0^1 \sqrt{1+x^4}\, dx$ *correct to two decimal places.*

Optional Exercises

Exercise 7. *Show that the Taylor series of the function* $f(x) = \frac{x}{1-x-x^2}$ *is* $\sum_{n=1}^{\infty} f_n x^n$ *where* f_n *is the* n*th Fibonacci number, that is,* $f_1 = 1, f_2 = 1$*, and* $f_n = f_{n-1} + f_{n-2}$ *for* $n \geq 3$*. By writing* $f(x)$ *as a sum of partial fractions and thereby obtaining the Taylor series in a different way, find an explicit formula for the* n*th Fibonacci number.*

Exercise 8. *Write down the Taylor series for* $\tan x$ *about the point* 0 *(this is much harder than the examples in Exercise 1).*

Exercise 9. *Can you construct a smooth (infinitely differentiable) function which takes the constant value* 0 *outside the interval* $[-1, 2]$ *and the constant value* 1 *on the interval* $[0, 1]$*.*

Exercise 10. *Prove the irrationality of the number* $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ *as follows. First show that* $e < 3$ *by comparing with a suitable geometric series. By Taylor's theorem (applied to* $a = 0$ *and* $b = 1$*) we know that*

$$e - \sum_{k=0}^{n} \frac{1}{n!} =: R_n = e^{\alpha} \frac{1}{(n+1)!}$$

for some α *between* 0 *and* 1*. Since* $e < 3$*,* $R_n < \frac{3}{(n+1)!}$*. Now suppose* e *is a rational number* c/d*, where* c *and* d *have no common factors. For* $n = d$*, we see that* $d!R_d$ *is an integer. On the other hand, using the estimate for* R_d *that we have obtained using Taylor's Theorem,* $d!R_d < \frac{d! \times 3}{(d+1)!} < 1$*, if* $d \geq 2$*.*
Try showing that π *is irrational using similar ideas.*

Tutorial sheet 4: Riemann integration

1. Let $f(x) = 1$ if $x \in [0,1]$ and $f(x) = 2$ if $x \in (1,2]$. Show from the first principles that f is Riemann integrable on $[0,2]$ and find $\displaystyle\int_0^2 f(x)dx$.

2. (a) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\displaystyle\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

   (b) Give an example of a Riemann integrable function on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\displaystyle\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a,b]$.

3. Evaluate $\displaystyle\lim_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:

   (i) $\displaystyle S_n = \frac{1}{n^{5/2}} \sum_{i=1}^{n} i^{3/2}$

   (ii) $\displaystyle S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2}$

   (iii) $\displaystyle S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}}$

   (iv) $\displaystyle S_n = \frac{1}{n} \sum_{i=1}^{n} \cos \frac{i\pi}{n}$

   (v) $\displaystyle S_n = \frac{1}{n} \left\{ \sum_{i=1}^{n} \left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n} \left(\frac{i}{n}\right)^{2} \right\}$

4. Compute

   (a) $\displaystyle\frac{d^2y}{dx^2}$, if $\displaystyle x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

   (b) $\displaystyle\frac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $\displaystyle F(x) = \int_1^{2x} \cos(t^2)dt$ (ii) $\displaystyle F(x) = \int_0^{x^2} \cos(t)dt$.

5. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\displaystyle\int_a^{a+p} f(t)dt$ has the same value for every real number a. (Hint : Consider $\displaystyle F(a) = \int_a^{a+p} f(t)dt,\ a \in \mathbb{R}$.)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t)dt.$$

   Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

7. Find the area of the region bounded by the given curves in each of the following cases.

(i) $\sqrt{x}+\sqrt{y}=1$, $x=0$ and $y=0$.

(ii) $y=x^4-2x^2$ and $y=2x^2$.

(iii) $x=3y-y^2$ and $x+y=3$.

8. Let $f(x)=x-x^2$ and $g(x)=ax$. Determine a so that the region above the graph of g and below the graph of f has area 4.5.

9. Find the area of the region inside the circle $r=6a\cos\theta$ and outside the cardioid $r=2a(1+\cos\theta)$.

10. Find the arc length of the each of the curves described below.

    (i) the cycloid $x=t-\sin t,\ y=1-\cos t,\ 0\le t\le 2\pi$.

    (ii) $y=\displaystyle\int_0^x \sqrt{\cos 2t}\,dt,\ 0\le x\le \pi/4$.

11. For the following curve
$$y=\frac{x^3}{3}+\frac{1}{4x},\ 1\le x\le 3,$$
find the arc length as well as the the area of the surface generated by revolving it about the line $y=-1$.

12. The cross sections of a certain solid by planes perpendicular to the x-axis are circles with diameters extending from the curve $y=x^2$ to the curve $y=8-x^2$. The solid lies between the points of intersection of these two curves. Find its volume.

13. Find the volume common to the cylinders $x^2+y^2=a^2$ and $y^2+z^2=a^2$.

14. A fixed line L in 3-space and a square of side r in a plane perpendicular to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution with L as the axis. Find the volume of the solid generated by this motion.

15. A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

Tutorial sheet 5: Functions of two variables, limits, continuity, partial derivatives

1. Find the natural domains of the following functions of two variables:

$$(i)\ \frac{xy}{x^2-y^2} \qquad (ii)\ \ln(x^2+y^2)$$

2. Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:
(i) $f(x, y) = x - y$ (ii) $f(x, y) = x^2 + y^2$ (iii) $f(x, y) = xy$

3. Using definition, examine the following functions for continuity at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero:

$$(i)\ \frac{x^3y}{x^6+y^2} \qquad (ii)\ xy\frac{x^2-y^2}{x^2+y^2} \qquad (iii)\ ||x|-|y||-|x|-|y|.$$

4. Suppose $f,\ g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x, y) \in \mathbb{R}^2$ are continuous:
(i) $f(x) \pm g(y)$ (ii) $f(x)g(y)$ (iii) $\max\{f(x),\ g(y)\}$ (iv) $\min\{f(x),\ g(y)\}$.

5. Let

$$f(x, y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2} \text{ for } (x, y) \neq (0, 0).$$

Show that the iterated limits

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] \ \&\ \lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right]$$

exist and both are equal to 0, but $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

6. Examine the following functions for the existence of partial derivatives at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero.

(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$

(ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$

7. Let $f(0, 0) = 0$ and

$$f(x, y) = (x^2 + y^2)\sin\frac{1}{x^2+y^2} \text{ for } (x, y) \neq (0, 0).$$

Show that f is continuous at $(0, 0)$, and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0, 0)$.

8. Let $f(0, 0) = 0$ and

$$f(x, y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{if } x \neq 0,\ y \neq 0 \\ x\sin 1/x, & \text{if } x \neq 0,\ y = 0 \\ y\sin 1/y, & \text{if } y \neq 0,\ x = 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0, 0)$ although f is continuous at $(0, 0)$.

9. Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ \ xy\frac{x^2-y^2}{x^2+y^2} \quad (ii)\ \ \frac{x^3}{x^2+y^2} \quad (iii)\ \ (x^2+y^2)\sin\frac{1}{x^2+y^2}$$

10. Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2}\ \text{if}\ y \neq 0.$$

Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

11. Show that the function $f(x,y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.

Tutorial Sheet 6: Tangent Planes, Maxima/minima, saddle points, Lagrange multipliers

1. Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$.

2. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin xy$ at the point $(1, 0)$ has the value 1.

3. Let $F(x, y, z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1, -1, 3)$ and the equations of the tangent plane and the normal line to the surface $F(x, y, z) = 7$ at $(1, -1, 3)$.

4. Find $D_{\underline{u}}F(2, 2, 1)$, where $F(x, y, z) = 3x - 5y + 2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$.

5. Given $\sin(x + y) + \sin(y + z) = 1$, find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y + z) \neq 0$.

6. If $f(0, 0) = 0$ and
$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0),$$
show that both f_{xy} and f_{yx} exist at $(0, 0)$, but they are not equal. Are f_{xy} and f_{yx} continuous at $(0, 0)$?

7. Show that the following functions have local minima at the indicated points.
(i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0, y_0) = (-1, 2)$

(ii) $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, \quad (x_0, y_0) = (0, 0)$

8. Analyze the following functions for local maxima, local minima and saddle points:
(i) $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$ (ii) $f(x, y) = x^3 - 3xy^2$

9. Find the absolute maximum and the absolute minimum of
$$f(x, y) = (x^2 - 4x)\cos y \text{ for } 1 \leq x \leq 3,\ -\pi/4 \leq y \leq \pi/4.$$

10. The temperature at a point (x, y, z) in 3-space is given by $T(x, y, z) = 400xyz$. Find the highest temperature on the unit sphere $x^2 + y^2 + z^2 = 1$.

11. Maximize the $f(x, y, z) = xyz$ subject to the constraints
$$x + y + z = 40 \text{ and } x + y = z.$$

12. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints
$$x + 2y + 3z = 6 \text{ and } x + 3y + 4z = 9.$$