January 27, 2014

Tutorial 3
(Closure, Interior, Countably infinite sets, bases)
MA 406 General Topology, IIT Bombay

(1) A topological space X is said to be first countable if every point in X has a countable neighbourhood base.
   (a) Show that every metric space is first countable.
   (b) Show that every second countable space is first countable.
   (c) Show that $\mathbb{R}_l$ is separable and first countable.

(2) A real number is said to be algebraic (over rationals) if it satisfies a polynomial equation of positive degree with rational coefficients. Show that the set of algebraic numbers is countable.

(3) Determine whether the following sets are countable.
   (a) The set of all functions $f : [0, 1] \to \mathbb{N}$.
   (b) The set of all finite subsets of $\mathbb{N}$.
   (c) The set of all functions $\mathbb{N} \to \mathbb{N}$.

(4) Let X be a topological space. Define the boundary (frontier) of a subset E of X as

$$Bd_X(E) = \overline{E} \cap \overline{X - E}.$$

   (a) Show that $\overline{E} = E \cup Bd_X(E)$.
   (b) Show that $E^\circ = E - Bd_X(E)$.
   (c) Show that $X = E^\circ \cup Bd_X(E) \cup (X - E)^\circ$.

(5) Let X be a metric space with metric ρ.
   (a) Show that the closure of a set E in X is given by $\overline{E} = \{y \in X \mid \rho(E, y) = 0\}$.
   (b) Show that the closed disk $U(x, \overline{\epsilon}) = \{y \mid \rho(x, y) \leq \epsilon\}$, is closed in X. Show that $U(x, \overline{\epsilon})$ may not be the closure of the disk $U(x, \epsilon) = \{y \mid \rho(x, y) < \epsilon\}$. (Hint : Let $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$.)

Definition 0.1. Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be topological spaces. Then define a topology on $X \times Y$ as the smallest topology containing the set $\{U \times V \mid U \in \mathscr{U}, V \in \mathscr{V}\}$.

(6) Show that $(A \times B)^\circ = A^\circ \times B^\circ$ in for subsets A and B of topological spaces X and Y respectively.

(7) Show that $\overline{A \times B} = \overline{A} \times \overline{B}$ for subsets A and B of topological spaces X and Y respectively.

(8) Let X be a topological space. Then $\mathscr{F} \subseteq \mathscr{P}(X)$ is a base for the closed sets in X if every closed set in X can be written as a intersection of elements of $\mathscr{F}$.
   (a) Show that $\mathscr{F}$ is a base of the closed sets in X if and only if the family $\mathscr{G} = \{A^c \mid A \in \mathscr{F}\}$ is a base for the topology on X.

(b) F is a base for the closed sets for some topology on X if and only if (a) whenever F_1 and F_2 belong to $\mathscr{F}$, $F_1 \cup F_2$ is the intersection of some elements of $\mathscr{F}$ and (b) $\cap_{F \in \mathscr{F}} F = \emptyset$.

(9) Let $\mathscr{U}$ be the topology generated by the collection $\{(a, b) \mid a, b \in \mathbb{R}\} \cup \{n \mid n \in \mathbb{N}\}$. Describe the interior operation in this collection .