Assignment 2 MA 401
Autumn 2018, IIT Bombay

(1) Define scalar multiplication by $\mathbb{R}$ on $\mathbb{R}^2$ as follows.

$$c(x,y) = (cx, 0).$$

Is this a vector space where addition is defined coordinatewise. ?

(2) Let $\mathrm{S}^1 \subset \mathbb{C}$ be defined as all complex numbers of unit length. Define $(\mathrm{S}^1, \star)$ to be $a \star b = ab$ which is product in $\mathbb{C}$. Define scalar multiplication with respect to $\mathbb{Q}$ as $\frac{m}{n} \circ x = x^{m/n}$. here choose the root with the smallest argument.

Is $(\mathrm{S}^1, \star, \circ)$ a vector space over $\mathbb{Q}$?

(3) Show that if F is a field of characteristic 0 then there is an injective function from $\mathbb{Q}$ to F.

(4) Is $(\mathbb{Z}, +)$ a $\mathbb{Q}$-vector space?

(5) Is $(\mathbb{Q}, +)$ a $\mathbb{R}$ vector space ?

(6) Let $(V, +, \circ)$ be a vector space over a field F and X be a set. Let $\mathcal{F}(X, V)$ denote the set of functions from X to V. Define the following operations for all $\alpha, \beta \in \mathcal{F}(X, V)$ and $k \in \mathrm{F}$;

$$(\alpha \underline{+} \beta)(a) = \alpha(a) + \alpha(b)$$

$$(k \underline{\circ} \alpha)(a) = k \circ \alpha(a)$$

Show that $(\mathcal{F}(X, V), \underline{+}, \underline{\circ})$ is a vector space.

(7) Let F be a field. Let $P_n(F) = \{a_0 + a_1 x + \ldots + a_n x^n \mid a_i \in F\}$ denote the set of all polynomials in F of degree less than or equal to n. Show that this is a vector space over F. Is it possible to think of it as a subspace of the space $\mathcal{F}(X, \mathrm{F})$?

(8) Show that the set of solutions to a n^{th} order real homogeneous linear differential equation is a subspace of $\mathcal{F}((0,1), \mathbb{R}))$.

(9) Let V_1 and V_2 be vector spaces over a field F then show that $(V_1 \times V_2)$ with coordinate wise addition and scalar multiplication is a vector space over F.

(10) Which of the following are subspaces of the given vector spaces?
  (a) $\{(0,0,0)\} \subseteq \mathbb{R}^3$.
  (b) $\{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 = 4, x_3 + x_4 = 2\} \subseteq \mathbb{R}^4\}$.
  (c) $\{a_0 + a_1 x + \ldots + a_n x^n \mid a_n \neq 0, a_i \in \mathrm{F}\} \subseteq P_n(\mathrm{F})$.
  (d) $\{A \in M_n(\mathrm{F}) \mid \sum_{i=1}^{n} a_{ii} = 0\} \subseteq M_n(\mathrm{F})$.
  (e) $\{A \in M_n(\mathrm{F}) \mid a_{ij} = a_{ji}\} \subseteq M_n(\mathrm{F})$.
  (f) Set of all continuous functions from $[0,1] \to \mathbb{R} \subseteq \mathcal{F}([0,1], \mathbb{R}]$.
  (g) Set of differentiable functions from $(0,1) \to \mathbb{R} \subseteq \mathcal{F}((0,1), \mathbb{R}]$.
  (h) Set of all real valued functions integrable over $[0,1] \subseteq \mathcal{F}((0,1), \mathbb{R}]$.