Assignment 3 MA 401
Autumn 2018, IIT Bombay

(1) For any subspaces W_1 and W_2 denote $W_1 + W_2 = \{w + w' \mid w \in W_1, w' \in W_2\}$.
  (a) Show that this is a subspace of V containing both W_1 and W_2.
  (b) Show that any other subspace of V containing both W_1 and W_2 must contain $W_1 + W_2$. What do you conclude?
  (c) If further $W_1 \cap W_2 = \{0\}$, then every element of $W_1 + W_2$ can written uniquely as a sum of elements in W_1 and W_2. (Note then $W_1 + W_2 := W_1 \oplus W_2$ and is called the direct sum of W_1 and W_2.)

(2) Show that the space $\mathcal{M}_n(F) = W_1 \oplus W_2$ where W_1 is subspace of all upper triangular matrices.

(3) Show that any linear transformation $T : \mathrm{F}^n \to \mathrm{F}^m$ can be represented by a $m \times n$ matrix A over F, that is, $T(x) = Ax$ where $x \in F^n$ is now considered as a $n \times 1$ matrix in F.

(4) Prove properties of $\mathcal{M}_{m\times n}(\mathrm{F})$.
  (a) $A(BC) = (AB)C$ for all $A \in \mathcal{M}_{m\times n}(\mathrm{F})$, $B \in \mathcal{M}_{n\times r}(\mathrm{F})$ and $C \in \mathcal{M}_{r\times k}(\mathrm{F})$.
  (b) $(A + B)C = AC + BC$ for all $A, B \in \mathcal{M}_{m\times n}(\mathrm{F})$, and $C \in \mathcal{M}_{n\times r}(\mathrm{F})$.
  (c) $C(D + E) = CD + CE$ for all $C \in \mathcal{M}_{m\times n}(\mathrm{F})$, and $D, E \in \mathcal{M}_{n\times r}(\mathrm{F})$.
  (d) $I_m A = A = AI_n$ where, $I_k \in \mathcal{M}_{k\times k}(\mathrm{F})$ is the identity matrix, that is, $(I_k)_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.

(5) State true or false with explanation.
  (a) The multiplication operation on $M_2(\mathrm{F}_2)$ is commutative.
  (b) If $A, B \in M_n(F)$ and AB is invertible then A and B are invertible.
  (c) If $T : U \to V$ and $S : V \to W$ are linear transformations of vector space over a field F, then $S \circ T$ is an isomorphism implies that T and S are isomorphisms.

(6) For all $A \in M_2(\mathrm{F}_2)$ give a criteria when A is invertible and write down its inverse.

(7) Show that $A \in \mathcal{M}_{n\times n}(F)$ is invertible if and only if the linear transformation $\mathrm{F}^n \to \mathrm{F}^n$ it represents is an isomorphism.

(8) Verify if the following are linear transformations. Check which of these are injective and/or surjective .
  (a) Define $T : \mathcal{M}_{2\times 3}(F) \to \mathcal{M}_{2\times 2}(F)$ for any field F as

$$T(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}) = \begin{pmatrix} a_{11} + a_{12} & a_{13} \\ a_{21}a_{22} & 0 \end{pmatrix}.$$

  (b) Define $T : \mathcal{M}_{2\times 3}(F) \to \mathcal{M}_{2\times 2}(F)$ for any field F as

$$T(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}) = \begin{pmatrix} a_{11} + a_{12} & a_{13} \\ a_{21} - a_{22} & a_{23} \end{pmatrix}.$$

(c) Define T to take a vector in $\mathbb{R}^2$ and map it to the vector in $\mathbb{R}^2$ obtained on rotating it by an angle θ .

(d) Define T from $\mathbb{R}^m$ to $\mathbb{R}^m$ to be the map which linearly translates a vector $v \in \mathbb{R}^m$ to $v + a \in \mathbb{R}^m$ for a fixed $a \in \mathbb{R}^m$.

(e) Define $T : \mathcal{P}_2(\mathrm{F}) \to \mathcal{P}_1(\mathrm{F})$ to be $T(a_0 + a_1 x + a_2 x^2) = a_0 x + a_2$.

(9) Are $\mathbb{R}^m$ and $\mathbb{R}^n$ isomorphic as vector spaces over $\mathbb{R}$? Why or why not? Answer the question only using what we have discussed in class.

(10) Let $\mathcal{P}(\mathbb{R})$ denote the space of all polynomials with real coefficients. Show that $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ defined as $T(f)(x) = \int_0^x f(t)\, dt$ is a linear transformation. Is it one-one? Is it onto? Explain your answer.