ASSIGNMENT 4 MA 401
AUTUMN 2018, IIT BOMBAY

(1) Solve the following system of equations and write down its solution set.
(a)

$$\begin{aligned} x_1 + x_2 + x3 &= 3 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

when $F = \mathbb{F}_5$.
(b)

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 1 \\ 3x_1 + 0.x_2 + 3x_3 &= 0 \end{aligned}$$

when $F = \mathbb{R}$.
(c)

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 0 \\ 3x_1 + 0.x_2 + 3x_3 &= 0 \end{aligned}$$

when $F = \mathbb{R}$.

(2) Find a system of linear equations over F with the following solution set if they exist. Show work.
(a) $\{(0, 0, 0)\}$ over $F = \mathbb{Q}$.
(b) $\{(1, 1), (2, 1)\}$ over $F = \mathbb{R}$.
(c) $\{(1, 1), (2, 1)\}$ over $F = \mathbb{F}_3$.
(d) $\{(s, s) \mid s \in F\}$ over $F = \mathbb{R}$.
(e) $\{(s, s + t) \mid s, t \in F\}$ over $F = \mathbb{R}$.

(3) Show that the space of all polynomials $\mathcal{P}(F) = \cup_n \mathcal{P}_n(F)$ is a vector space with coeffcientwise addition and scalar multiplication.

(4) Check if the following subsets W of the given vector space V are subspaces of V.
(a)

$$W = \{(a, b, c) \in \mathbb{R}^3 \mid a + 3c + b = 0\} \subset \mathbb{R}^3 = V.$$

(b)

$$W = \{f : \mathbb{F}_3 \to \mathbb{F}_3, f(x) = a + ax + a^2x^2 \mid a \in \mathbb{F}_3\} \subset \mathcal{P}_2(\mathrm{F}_3) = V.$$

(c)

$$W = \{\begin{bmatrix} s & 3t \\ s + t & 0 \end{bmatrix} \mid s, t \in \mathbb{C}\} \subset \mathcal{M}_{2\times 2}(\mathbb{C}) = V.$$

(d) Let F be a field.

$$W = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in F, A^2 = A\} \subset \mathcal{M}_{2\times 2}(F) = V.$$

(5) Which of the following functions are linear transformations?

(a) Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ be defined as

$$T(f) = \begin{pmatrix} f(0) & f(1) \\ f(2) & f(3) \end{pmatrix}.$$

(b) Let $T : \mathcal{M}_{2\times 3}(\mathbb{Q}) \to \mathcal{M}_{3\times 2}(\mathbb{Q})$ as

$$T(A) = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix}.$$

(c) Let $T : \mathcal{C}(\mathbb{R}, \mathbb{R}) \to \mathcal{C}(\mathbb{R}, \mathbb{R})$ be defined as $T(f)(x) = xf(x)$.

(6) Show that the set of all odd functions $V_o = \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x)\}$ and $V_e = \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x)\}$ are subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$

Prove that $V_o \oplus V_e = \mathcal{F}(\mathbb{R}, \mathbb{R})$.

(7) Show that the following statements are equivalent for a vector space V and $\mathcal{B} \subseteq V$.

(a) The set $\mathcal{B}$ is the minimal spanning subset of V.
(b) The set $\mathcal{B}$ is linearly independent and spans V.
(c) The set $\mathcal{B}$ is a maximal linearly independent subset of V.

(8) Check if the following sets are linearly independent or not.

(a) $\{(x^i) \mid i \in \mathbb{N}\}$ where $(x^i)_n = \begin{cases} 1 & n = i, i+1 \\ 0 & \text{else} \end{cases}$
(b) $\{x - 25, x + 5, x^2 + 3x + 1\} \subset \mathcal{P}_2(\mathbb{R})$.
(c) $\{(2, 1, 1, 1), (3, 3, 3, 3), (3, 2, 1, 1)\} \subset \mathrm{F}_5^4$.

(9) Find a basis for $\mathcal{P}(\mathbb{R})$, the space of all polynomials of finite degree with real coeffcients.

(10) Describe the span of the following sets in the respective vector spaces.

(a) $\{(x^i) \mid i \in \mathbb{N}\} \subseteq \mathcal{F}(\mathbb{N}, \mathbb{R})$ where $(x^i)_n = \begin{cases} 1 & n = i, i+1 \\ 0 & \text{else} \end{cases}$
(b) $\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\} \subset \mathcal{M}_{2\times 2}(\mathbb{Q})$.
(c) $\{f(x) = 5, g(x) = e^x, h(x) = x^2\} \subseteq \mathcal{C}(\mathbb{R}, \mathbb{R})$.
(d) $\{p_1(x) = 55, p_2(x) = x^2 + x\} \subseteq \mathcal{P}(\mathbb{R})$.