Assignment 5 MA 401
Autumn 2018, IIT Bombay

(1) Let V be a vector space over a field F. State true or false with explanation.
   (a) If $\{v_1, v_2, v_3, v_4\}$ is a basis of V then $\{v_1, v_2, v_3\}$ spans V.
   (b) If $\{v_1, v_2, v_3, v_4\}$ is a basis of V then $\{v_1, v_2, v_5\}$ is linearly independent for any $v_5 \in V$ which is not equal to v_1 and v_2.
   (c) If T is a linearly independent subset of V then every subset $S \subseteq T$ is linearly independent in V.
   (d) If T spans V then every set S containing T, that is , $T \subseteq S$ spans V.

(2) Let V and W be a vector spaces over a field F with a linear transformation $T : V \to W$.
   (a) Let T be injective. If $S \subseteq ofV$ is a linearly independent set then $T(S)$ is a linearly independent set in W.
   (b) Show that for the previous statement to be true it is essential that T be injective.
   (c) Let T be surjective. Then if $S \subseteq V$ is a spanning set of V Then $T(S)$ is a spanning set of W.
   (d) Show that for the previous statement to be true it is essential that T is surjective.
   (e) Show that T is an isomorphism if and only if for any basis $\mathcal{B}$ of V, $T(\mathcal{B})$ is a basis of W.

(3) Let V and W be a vector spaces over a field F with a linear transformation $T : V \to W$.
   (a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.
   (b) Prove that if $\dim(V) > \dim(W)$ then T cannot be one-one.

(4) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathrm{Ker}(T) = \mathrm{Im}(T)$.

(5) Define $T, U : \mathcal{F}(\mathbb{N}, \mathbb{R})] \to \mathcal{F}(\mathbb{N}, \mathbb{R})$ as $T(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots)$ and $U((a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, a_4, \ldots)$.
   (a) Show that T and U are linear transformations.
   (b) Show that T is onto but not one-one.
   (c) Show that U is one-one but not onto.

(6) Let V be a finite dimensional vector space and $T : V \to V$ be linear.
   (a) If $V = \mathrm{Ker}(T) + \mathrm{Im}(T)$, show that $V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$.
   (b) If $\mathrm{Ker}(T) \cap \mathrm{Im}(T) = \{0\}$ show that $V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$.
   (c) Show that for both the above parts its essential that V be finite dimensional.
   (d) Show that $\dim V = \dim \mathrm{Ker} T + \dim \mathrm{Im} T$.

(7) Let E_n denote the standard ordered basis for $\mathbb{R}^n$. Verify that $\mathcal{B} = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$ is a basis for $\mathbb{R}^3$. Let $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Find $[T]_{E_2}^{\mathcal{B}}$.

(8) Let $E_{m\times n}$ define the standard basis for $\mathcal{M}_{m\times n}(\mathbb{R})$. Verify $\mathcal{B} = \{\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\}$ is a basis of $\mathcal{M}_2(\mathbb{R})$.

Let $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(R)$ be given by $T(A) = A^t$. Find $[T]_{\mathcal{B}}^{E_{2\times 2}}$.