Assignment 6 MA 401
Autumn 2018, IIT Bombay

(1) Let A be a $m \times n$ matrix. Let T_A denote the linear transformation represented by A. Let A' be the matrix obtained by on applying an elementary row operation to A.

Show that $T_{A'} : F^n \to F^m$ is represented by EA where E is the $m \times m$ matrix obtained by applying the same elementary transformation to the identity matrix.

(2) Show that if an $m-1 \times n-1$ matrix B' can be transformed by elementary operations (row and column) to an $m-1 \times n-1 D'$ then the $m \times n$ matrix $B = \begin{pmatrix} 1 & 0 \\ 0 & B' \end{pmatrix}$ can be transformed to the $m \times n$ matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & D' \end{pmatrix}$

(3) Let A be a $m \times n$ matrix.
  (a) Show that if A' is obtained from A be a elementary row or column operation then the rank does not change.
  (b) Let A have rank r. Induct on the number of rows to show that A can transformed by elementary row operations to $[e_1 \dots e_r \; 0 \dots 0]$.
  (c) Show that if A is a $n \times n$ matrix with rank n then it can be written as a product of elementary matrices.
  (d) Let A be an $m \times n$ matrix. Show that there exist invertible $m \times m$ matrix X and invertible $n \times n$ matrix Y such that

$$A = X[e_1 \dots e_r \; 0 \dots 0]Y.$$

(4) Show that $\det A = \det A^t$.

(5) Let V be a vector space be a vector space over F. Let $V^* = \mathcal{L}(V, F)$ denote the dual vector space over F. Let V and W be finite dimensional vector spaces over F with bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. Let $T : V \to W$ be a linear transformation.
  (a) Show that a basis $\mathcal{B}$ of V gives a basis $\mathcal{B}^*$ of V^* and $\dim V = \dim V^*$
  (b) Show that T induces a linear transformation $T^* : W^* \to V^*$.
  (c) Show that $[T^*]_{\mathcal{B}'^*}^{\mathcal{B}^*} = ([T]_{\mathcal{B}}^{\mathcal{B}'})^t$.

(6) State true or false with explanation.
  (a) The rank of a matrix is equal to the number of linearly independent rows of a matrix.
  (b) The function $\delta : M_n(F) \to F$ defined as $\delta(A) = 0$ for all $A \in M_n(F)$ is a determinant function.
  (c) A determinant function is a linear transformation.

(7) Which of the following functions $\delta : M_3(F) \to F$ are multilinear?
  (a) $\delta(A) = 1$ for all A
  (b) $\delta(A) = a_{22}$ for all A.
  (c) $\delta(A) = a_{11}a_{21}a_{32}$ for all A.

(8) Show that $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}.\det(\hat{A_{ij}})$ where $\hat{A}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and jth column.

(9) Show that $\det A = \sum_{j=1}^{n} (-1)^{j+1} a_{ij}.\det(\hat{A_{ij}})$ is multilinear and alternating function on the rows of A.

(10) Prove the adjoint formula for inverse of a matrix and compute the inverse of the matrix.

$$\begin{pmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{pmatrix}.$$