Assignment 7 MA 401
Autumn 2018, IIT Bombay

(1) Show that the characteristic polynomial of the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{k-1} \end{pmatrix}$$

is $(-1)^k(a_0 + a_1 t + \ldots + a_{k-1}t^{k-1} + t^k)$.

(2) Is $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ defined as $T(f(x) = f(0)+f(1)(x+x^2)$ diagonalizable? If yes, find the basis which diagonalizes the representing matrix.

(3) Suppose that $A \in M_{n\times n}(F)$ has two distinct eigenvalues λ_1 and λ_2 with $\dim W_{\lambda_1} = n - 1$. Show that A is diagonalizable.

(4) Let $T : V \to V$ be an isomorphism. Show that T is diagonalizable if and only if T^{-1} is diagonalizable.

(5) Let $T : V \to V$ be a linear transformation on a n-dimensional vector space V. Show that if $T^k = 0$ for some k then $T^n = 0$.

(6) Find a 3×3 matrix whose minimal polynomial is t^3.

(7) Determine whether for $T : V \to V$ the given subspace W is T invariant.
  (a) Let $V = \mathcal{C}[0, 1]$ and $T(f(t)) = (\int_0^1 f(x)\, dx)\, t$ and
  $$W = \{f \in V \mid f(t) = at + b \text{ for some } a \text{ and } b\}.$$
  (b) $V = \mathcal{M}_{2\times2}(\mathbb{R})$, $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$ and $W = \{A \in V \mid A^t = A\}$.

(8) Find the T cyclic subspace generated by $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ where $T : \mathcal{M}_{2\times2}(\mathbb{R}) \to \mathcal{M}_{2\times2}(\mathbb{R})$ is defined as $T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$.

(9) Show that if $T : V \to V$ is a linear transformation and W is a T-invariant subspace then $\overline{T} : V/W \to V/W$ defined as $\overline{T}(\overline{v}) = \overline{T(v)}$ is a linear transformation.

(10) If f, g and h are characteristic polynomials of T, $T|_W$, and $\overline{T}$ respectively . Prove that $f(t) = g(t)h(t)$.

(11) Show that if T is diagonalizable then $\overline{T}$ is diagonalizable.

(12) Show that if $T|_W$, and $\overline{T}$ are diagonalizable and **they have no common eigenvalues** then so is T.

(13) Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}$. Let W be the T_A cyclic subspace of $\mathbb{R}^3$ generated by e_1.

(a) Compute the characteristic polynomial of $(T_A)_{|W}$.

(b) Show that $e_2 + W$ is a basis for $\mathbb{R}^3/W$ and use this to compute the characteristic polynomial of T.