Assignment 8 MA 401
Autumn 2018, IIT Bombay

(1) The following sequence of exercise shows that if $T : V \to V$ is a linear transformation on a finite dimensional vector space whose characteristic polynomials splits then $K_\lambda = \{v \in V \mid (T - \lambda I)^r v = 0 \text{ for some } r\}$ has a basis consisting of $(T - \lambda I)$ cycles.
   (a) Show that if $\dim K_\lambda = 1$ then statement is true.
   (b) Show that $K_\lambda$ is $(T - \lambda I)$ invariant and note the $\mathrm{Ker}(T - \lambda I)$ is exactly the eigenspace of $\lambda$ $W_\lambda$ in $V$.
   (c) Assume statement is true when $\dim K_\lambda < n$. Let $\dim K_\lambda = n$. Show that $K_\lambda \cap \mathrm{Im}(T - \lambda I)$ has dimension $< n$ and satisfies the hypothesis of the theorem . Therefore, a union of $(T - \lambda I)$ cycles form a basis $\mathcal{B}'$ for $K_\lambda \cap \mathrm{Im}(T - \lambda I)$ .
   (d) Find a basis of $W_\lambda$ which is an extension of the set of linearly independent vectors which are initial vectors of cycles in $\mathcal{B}'$.
   (e) Now construct an appropriate basis of cycles for $K_\lambda$ to prove the theorem.

(2) Find the Jordan canonical basis for the following linear transformations.
   (a) Let $T$ be a linear transformation on $\mathcal{P}_2(\mathbb{R})$ defined by $T(f(x)) = 2f(x) - f'(x)$.
   (b) Let $V = \mathrm{Span}\{1, t, t^2, e^t, te^t\} \subseteq \mathcal{C}([0, 1])$ and $T$ is a linear operator on $V$ defined by $T(f) = f'$.
   (c) Let $T$ be a linear operator on $\mathcal{M}_{2\times 2}(\mathbb{R})$ such that $T(A) = 2A + A^t$ for all $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$.

(3) Let $p$ be the minimal polynomial and $f$ the characteristic polynomial of a linear transformation $T : V \to V$ where $V$ is a finite dimensional vector space over a field $F$. Show that if $p$ splits then so does $f$. (Hint: You can use that every field is contained in an algebraically closed field )

(4) Let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Find a $T_A$ cyclic subspace of $\mathbb{R}^2$ which is all of $\mathbb{R}^2$. Write the transformation in terms of a basis of this $T_A$-cyclic subspace. Try to generalize this idea to explain how to get a nice form when the characteristic polynomial does not split but consists of irreducible factors. More precisely, try to show that if $\phi(t)$ is a irreducible factor of the characteristic polynomial then $K_\phi = \{v \in V \mid \phi(T)^k(v) = 0\}$ is a non-zero subspace of $V$ and that $V = \oplus_i K_{\phi_i}$ where $\phi_i$ are the irreducible factors in the characteristic polynomial.

(5) Let $T : V \to V$ be a linear transformation where $V$ is finite dimensional.

(a) Let the characteristic polynomial of $T$ split. Then $T$ is nilpotent if and only if its characteristic polynomial is of the form $t^n$.

(b) If $T$ is nilpotent show that its characteristic polynomial is of the form $t^n$. *Prove this statement without using the idea of minimal polynomials.* (**Hint:** Use induction. and consider the $T$-invariant subspace $\mathrm{Im}(T)$.)

(6) Prove that any upper triangular square matrix with all diagonal entries 0 is nilpotent.

(7) Let $T : V \to V$ be a linear transformation where $V$ is finite dimensional such that its characteristic polynomial splits with $\lambda_1, \ldots, \lambda_k$ the distinct eigenvalues.

(a) Show that every vector in $v \in V$ can be written uniquely as a sum of vectors in $v_i \in K_{\lambda_i}$, $i = 1, 2 \ldots k$.

(b) Let $S : V \to V$ be defined as $S(v) = \sum_{i=1}^{k} \lambda_i v_i$. Show that $S$ is diagonalizable linear transformation.

(c) Define $U : V \to V$ as $U = T - S$. Show that $U$ is nilpotent and $SU = US$.

(d) Prove the converse that is, if $T$ can be written as $S + U$ where $S$ is diagonalizable and $U$ is nilpotent and $SU = US$, then $T$ and $S$ have the same characteristic polynomial and $S(x) = \lambda_1 x_1 + \ldots + \lambda_k x_k$ where $\lambda_i$'s are eigenvalues and $x_i$'s are as defined earlier. ( *Hint: First show that $T$ and $S$ have the same eigenvalues and eigenvectors of $S$ are generalized eigenvectors of $T$. Then use that $S$ is diagonalizable to show that $V$ can be written as a direct sum of generalized eigenspaces.)*

(8) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by a matrix $\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$. Show that there exists diagonalizable and nilpotent lienar transformations such that $T = D + N$ and $DN = ND$.