ASSIGNMENT 9 MA 401
AUTUMN 2018, IIT BOMBAY

(1) Let $V = \mathbb{C}^3$ with standard inner product space. Let $x = 2, 1+i, i)$, $y = (2-i, 2, 1+2i)$. Compute $< x, y >$, $\|x\|$, $\|y\|$ and $\|x+y\|^2$.

(2) Show the following identities. If V is a real inner product space then

$$< v, w >= \frac{1}{4}\|v+w\|^2 - \frac{1}{4}\|v-w\|^2$$

and if V is a complex inner product space then

$$< v, w >= \frac{1}{4}\|v+w\|^2 - \frac{1}{4}\|v-w\|^2 + \frac{i}{4}\|v+iw\|^2 - \frac{i}{4}\|v-iw\|^2.$$

(3) Let $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$. Show that $< x, y >= \mathrm{tr}(xAy^*)$ is a inner product on $\mathbb{C}^2$.

(4) Show that if V is an inner product space then $d(x, y) = \|x - y\|$ for all $x, y \in V$ defines a metric on V.

(5) Consider the standard inner product on $\mathcal{P}_2(\mathbb{R}) \subset \mathcal{C}(0, 1)$. Find a orthonormal basis for it.

(6) Find the orthogonal complement of span of $S = \{(1, i, 0), (1-i, 2, 4i)\}$.

(7) Show that if $T : V \to V$ is a linear transformation then $\mathrm{Im}(T)^{\perp} = \mathrm{Ker}(T^*)$.

(8) Give an example to show that
   (a) not every normal linear transformation on a finite dimensional real inner product space is diagonalizable.
   (b) not every a normal operator on a infinite dimensional complex inner product space is diagonalizable.

(9) Let T be a finite dimensional inner product space V. If $< T(x), y >= 0$ for all $x, y \in V$ then show that $T = 0$.

(10) Find the adjoint of $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ defined as $Tf = f' + 3f$.

(11) Find the value of $T * (3-i, 1+2i)$ where $T : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$T(w, z) = (2w + iz, (1-i)w).$$

(12) State true or false with explanation.
  (a) Every diagonalizable operator is self adjoint.
  (b) Every self adjoint operator is normal.
  (c) Every normal operator on a complex inner product space is self adjoint.
  (d) Orthogonal complement of any set of vectors is a subspace.
  (e) Let $T : V \to V$ be a linear transformation of finite dimensional inner product spaces. Then given any basis $\mathcal{B}$ of V, $[T^*]_{\mathcal{B}} = ([T]_{\mathcal{B}})^*$.

(13) If T is a self adjoint linear transformation from $V \to V$ where V is a finite dimensional inner product space and $< x, Tx >= 0$ for all $x \in V$. Then $T = 0$.

(14) Let T be a linear transformation on a complex inner product space.
  (a) If T is self adjoint then $< T(x), x >$ is real for all $x \in V$.
  (b) If T satisfies $< T(x), x) >= 0$ for all $x \in v$ then $T = 0$. (Hint: Expand by replacing x by $x + y$ and $x + iy$.)
  (c) If $< T(x), x >= 0$ is real for all $x \in V$ then $T = T^*$.

(15) If T is a self adjoint linear transformation from $V \to V$ where V is a finite dimensional inner product space and $< Tx, x >= 0$ for all $x \in V$. Then T