PARTIAL SOLUTIONS TO TUTORIAL I

MA 406 GENERAL TOPOLOGY, IIT BOMBAY

(1)(a) $\cup \mathcal{U}_\alpha$ is not a topology in general.

Counter example - use (2).

(1)(b). The unique smallest topology containg all the $\mathcal{U}_\alpha$'s is

$$\mathcal{F} = \bigcap \{ \mathcal{U} \mid \mathcal{U} \supseteq \mathcal{U}_\alpha \ \forall \alpha, \mathcal{U} \text{ is a topology } X \}$$

* Show that this is smallest & unique.

(That $\mathcal{F}$ is a topology follows from 1(a)).

The largest topology contained in all the $\mathcal{U}_\alpha$'s will be $\bigcap \mathcal{U}_\alpha$.

Again this is a topology by 1(a)

* Show that it is largest topology contained in $\mathcal{U}_\alpha \ \forall \alpha$ & unique!

Remark: Explain why smallest, largest etc do not imply unique directly?

(3)(a). $E \cup Bd_X(E) = E \cup (\bar{E} \cap \overline{X-E})$

$= (E \cup \bar{E}) \cap (E \cup \overline{X-E})$

(since $E \subseteq \bar{E}$) $= \bar{E} \cap \left(E \cup \left(\bigcap_{\substack{X-E \subseteq F \\ F \text{ closed}}} F\right)\right)$

$= \bar{E} \cap \left(\bigcap_{\substack{X-E \subseteq F \\ F \text{ closed}}} (E \cup F)\right)$

But $X-E \subseteq F \Rightarrow X = E \cup X-E \subseteq E \cup F \subseteq X$

$\Rightarrow E \cup F = X$.

$\therefore E \cup Bd_X(E) = \bar{E} \cap (\bigcap X)$

$= \bar{E} \cap X = \bar{E}$

3(b). $E^{\circ} \cup Bd_X(E) = E^{\circ} \cup (\bar{E} \cap \overline{X-E})$

$= (E^{\circ} \cup \bar{E}) \cap (E^{\circ} \cup \overline{X-E})$

Note. $\overline{X-E} = (E^{\circ})^{c}$

$\Rightarrow E^{\circ} \cup Bd_X(E) = (E^{\circ} \cup \bar{E}) \cap (E^{\circ} \cup (E^{\circ})^{c})$

$= \bar{E} \cap X = \bar{E}$.

16 c)

The answer follows from previous fact.

(4). ~~Check that the $\phi, \mathbb{R}^2 \in \mathcal{U}_{RD}$ where~~

Let $\mathcal{U}_{RD}$ consists of radially open sets in $\mathbb{R}^2$.

We ~~only~~ need to show that this defines a topology on $\mathbb{R}^2$.

* The important one is to check that the intersection of two radially open sets is radially open.

Let A and B be radially open.

Let $x \in A \cap B$. There exists a line segment through x in every radial direction in both A and B. Pick the least radius in each direction then. There will be a radial nbd around x in $A \cap B$.

*. Show that $\mathcal{U}_{(\mathbb{R}^2, \|\|_2)} \subseteq \mathcal{U}_{RD}$.

where $\mathcal{U}_{(\mathbb{R}^2, \|\|_2)}$ is the usual topology on $(\mathbb{R}^2, \|\|_2)$.

* Show that ~~$\mathcal{U}_{RD}$~~ $\mathcal{U}_{(\mathbb{R}^2, \|\ \|_2)} \subsetneq \mathcal{U}_{RD}$.

Hint: Take a open disk around $(0,0)$ of radius 1.

~~Re~~ Since rational angles are countable denote them as $\theta_1, \theta_2, \ldots$

Remove a point at distance $1/n$ from $(0,0)$ in direction θ_n for all n, & call this new set A.

Show that this set A is not open in $(\mathbb{R}^2, \|\ \|_2)$ but is radially open.

(5) Show that the given collection of sets is has the two properties of a base.

* Why is this sufficient to show that it is a topology.