TUTORIAL 2. PARTIAL SOLUTIONS

Q1: Let X be a topological space. Then $\mathcal{F} \subseteq P(X)$ is a base for the closed sets in X if every closed set in X can be written as an intersection of elements of $\mathcal{F}$.

(a). Show that $\mathcal{F}$ is a base of the closed sets in X iff the family $\mathcal{G} = \{A^c \mid A \in \mathcal{F}\}$ is a base for the topology on X.

Ans: Follows from defn

(b). Show that $\mathcal{F}$ is a base ~~of X~~ closed sets for some topology in X iff (a) whenever $F_1, F_2 \in \mathcal{F}$, $F_1 \cup F_2$ is the intersection of some elements of $\mathcal{F}$ and (b) $\bigcap_{F \in \mathcal{F}} F = \phi$.

Ans: $\mathcal{F}$ is a base of closed sets for some topology on X iff $\mathcal{G} = \{A^c \mid A \in \mathcal{F}\}$ is a base for the topology.

iff every $B_1 = A_1^c \in \mathcal{G}$,

& $B_2 = A_2^c \in \mathcal{G}$,

if $B_1 \cap B_2 \neq \phi$ Then $B_1 \cap B_2 = \cup B_j$ for some $B_j \in \mathcal{G}$

and $\displaystyle\bigcup_{B \in \mathcal{G}} B = X$.

iff. where $B_i = A_i^c$ for some $A_i \in \mathcal{F}$.

iff $A_1^c \cap A_2^c \neq \phi \Rightarrow A_1^c \cap A_2^c = \displaystyle\bigcup_{A_j \in \mathcal{F}} A_j^c$

and $\displaystyle\bigcup_{A \in \mathcal{F}} A^c = X$.

iff $X - A_1 \cup A_2 \neq \phi \Rightarrow (A_1 \cup A_2)^c = (\cap A_j)^c$ for some $A_j \in \mathcal{F}$

and $\Big(\displaystyle\bigcap_{A \in \mathcal{F}} A\Big)^c = X$

iff $A_1 \cup A_2 \neq X \Rightarrow A_1 \cup A_2 = \cap A_j$ for some $A_j \in \mathcal{F}$

and $\displaystyle\bigcap_{A \in \mathcal{F}} A = \phi$

(2) A real number is algebraic (over rationals) if it satisfies a polynomial equation of positive degree with rational ~~coeff~~ coefficient. Show that the set of algebraic numbers is countable.

Ans: Any rational polynomial is of the form. $a_n x^n + \dots + a_0$

In fact, $\mathbb{Q}[x] = \{a_n x^n + \dots + a_0 \mid a_0, \dots, a_n \in \mathbb{Q}, n \in \mathbb{N}\}$

is a countable set. why? show this!

Now use the fact that a polynomial over $\mathbb{Q}$ can have only finitely many roots.

Set of alg numbers = roots of rational polynomials.

Show now this has to be countable.

(3) Determine whether the following sets are countable.

(a) The set of all functions $f : [0,1] \to \mathbb{N}$

<u>Ans</u>: let $A = \{ f : [0,1] \to \mathbb{N} \mid f \text{ is a function} \}$

~~Then~~. Define $f_x : [0,1] \to \mathbb{N}$ for every $x \in [0,1]$

$$\text{as } f_x(y) = 1 \quad \text{if } x \neq y$$
$$= 2 \quad \text{if } x = y.$$

Then. $f_x = f_y$ if and only if $x = y$

$\Rightarrow B = \{ f_x \}_{x \in [0,1]} \subseteq A$.

But $\phi : [0,1] \to A$

$$\phi(x) = f_x$$

defines a injective function (Verify this!)

whose image $\phi([0,1]) = B$

$\Rightarrow$ B is uncountable & $B \subseteq A$.

$\Rightarrow$ A is uncountable as otherwise B should have been countable (subsets of countable sets is countable)

(b) The set of all finite subsets of $\mathbb{N}$

Ans. Claim: A = Set of all finite subsets of $\mathbb{N}$

$$= \bigcup_{k \in \mathbb{N}} P(\{1,2,\dots,k\})$$

where $P(X)$ denotes the power set of X.

<u>Reason</u>: Let X be any finite subset of $\mathbb{N}$. Then $X \to \{1,2,\dots,m\}$ is a bijection for some m.

$\Rightarrow$ X has a largest element.

(Because if not then choose least element $m_1 \in X$ & define $f(m_1) = 1$, ~~$\exists\, m_2 > m_1$ s.t~~ $m_2 \in X$. Now define m_k = least $X - \{m_1,\dots,m_{k-1}\}$ & $f(m_k) = k$

This

If X has no largest element $\Rightarrow$ $X - \{m_1, \dots, m_{k-1}\}$ is non empty for all k! $\Rightarrow$ X is in bijection with $\mathbb{N}$, which contradicts finiteness of X).

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$\therefore$ X has a largest element say m.

$\Rightarrow \quad X \subseteq \{1, 2 \ldots, m\}$.

$\Rightarrow \quad X \in \mathcal{P}(\{1, 2 \ldots, m\})$

$\therefore \quad A \subseteq \bigcup\limits_{k \in \mathbb{N}} \mathcal{P}(\{1, 2 \ldots, k\})$.

Also every element of $\bigcup\limits_{k \in \mathbb{N}} \mathcal{P}(\{1, 2 \ldots, k\})$

is a finite subset of $\mathbb{N}$

$\therefore \quad \bigcup\limits_{k \in \mathbb{N}} \mathcal{P}(\{1, 2 \ldots, k\}) = A$.

Now. $\mathcal{P}(\{1, 2 \ldots, k\})$ is finite i.e countable

& we are taking ctbl union of

ctbl sets $\Rightarrow$ A is ctbl.

(c). The set of all functions $\mathbb{N} \to \mathbb{N}$

<u>Ans</u>: Let A = set of all functions $\mathbb{N} \to \mathbb{N}$

~~Then B = set of all functions~~ $\mathbb{N} \to \{0, 1\}$

~~is a subset of A,~~

~~Claim: B is uncountable.~~

~~Reason: Note.~~

3(c) Ans. Let B = Set of all functions $f : \mathbb{N} \to \{1, 2, \dots, 10\}$

$= $ Set of all functions $g : \mathbb{N} \to \{0, 1, \dots, 9\}$.

~~where~~ since $\{0, 1, \dots, 9\} \to \{1, 2, \dots, 10\}$ is a bijection.

Then $B \subseteq A$.

Now we claim that B is in bijection with $[0,1]$.

Reason: Define $\phi : B \to [0,1]$ as

$$\phi(f) = 0.f(1)f(2)\dots$$

Then verify ϕ is a surjection and that this $\Rightarrow$ B is not countable

$\Rightarrow$ A is not countable.

(4). A metric space is said to be separable if it has a countable dense set. Show that every separable space is second countable.

Ans: Let D be a countable dense set.

~~Then~~ let $D = \{a_1, a_2, \ldots\}$.

Show that $\{B_d(a_i, 1/n) \mid n \in \mathbb{N}, i \in \mathbb{N}\}$

defines a base for X.

5) Show that $S = \{(-\infty, a), (b, \infty) \mid a, b \in \mathbb{R}\}$ forms a subbase for the usual topology on $\mathbb{R}$.

Ans: Verify that every open interval can be written as a finite intersection of elements of S.

6) Describe the topology generated by the collection $\{(a, \infty) \mid a \in \mathbb{R}\}$.

Ans: Note that $\bigcup_{a \in \mathbb{R}} (a, \infty) = \mathbb{R}$ and for any

$(a, \infty) \cap (b, \infty) = (c, \infty)$ where $c = a$ if $a \geq b$
$= b$ if $b \geq a$.

$\therefore$ The topology gen by the collection

$\{\bigcup_{i \in I} (a_i, \infty)$ I is any indexing , $a_i \in \mathbb{R}\}$

But $\bigcup_{i \in I} (a_i, \infty) = (\inf_{i \in I} a_i, \infty)$ if inf exists

$= \mathbb{R}$ else.

$\therefore$ The topology is in fact $\{ [a, \infty) \mid a \in \mathbb{R} \}$.

7) Let $\mathcal{U}$ be the topology generated by the collection $\{ \{(a,b) \mid a, b \in \mathbb{R}\} \} \cup \{ \{n\} \mid n \in \mathbb{N} \}$

Describe the interior operation in this collection

Ans: ~~First show that the un~~

Let $\mathcal{B} = \{ (a,b) \mid a, b \in \mathbb{R} \} \cup \{ \{n\} \mid n \in \mathbb{N} \}$

Show that $\bigcup_{A \in \mathcal{B}} A = \mathbb{R}$ and for any

$A_1, A_2 \in \mathcal{B}$, $\phi \neq A_1 \cap A_2$ ~~contains~~ $\Rightarrow \exists A_3 \in \mathcal{B}$

s.t. $A_3 \subseteq A_1 \cap A_2$.

(Consider A_1, A_2 of the form (a_i, b_i), of the form $\{n_i\}$, etc.).

The interior of a set A is the largest open subset in A.

Here open sets are unions of open intervals and natural numbers.

Show that $\text{int} A = \text{int}_{\mathbb{R}} A \cup (A \cap \mathbb{N})$

where $\text{int}_{\mathbb{R}} A$ is the interior of A in usual topology on $\mathbb{R}$.

(8) Let $\mathcal{U} = \{ A \cup B \mid A$ is open $(\mathbb{R}, |\,|)$ and $B \subseteq \mathbb{R} - \mathbb{Q} \}$.

Show that $\mathcal{U}$ is a topology on $\mathbb{R}$ and describe neighbourhood base at all rational & irrational points.

Ans. It is easy to verify it is a topology.

Claim 1: If $x \in \mathbb{R} - \mathbb{Q}$, $\{\{x\}\}$ is a neighbourhood base at x.

Claim 2: If $x \in \mathbb{Q}$, $\{ (x - 1/n, x + 1/n) \mid n \in \mathbb{N} \}$ is a neighbourhood base at x.

Verify the claims to solve the problem

(9) This was discussed in class.