PARTIAL SOLUTIONS TO TUTORIAL 3.

Ans
(1) T (a). Verify that $\mathcal{G}_x = \{[x,b) \mid b > x, b \in \mathbb{R}\}$ satisfies the properties required for it to form a neighbourhood base of some topology.

Namely, we note

(1) For any $[x,b) \in \mathcal{G}_x$. Clearly $x \in [x,b)$

(2). For any $[x,b_1)$ and $[x,b_2) \in \mathcal{G}_x$.

$[x,b_1) \cap [x,b_2) = [x,b)$ where $b = b_1$ if $b_1 < b_2$

$b = b_2$ if $b_2 < b_1$.

(3). Let $W = [x,b) \in \mathcal{G}_x$. We want to find $V \subseteq W$ s.t for every $a \in V$ $\exists\ [a,b_a) \in \mathcal{G}_b$ s.t $[a,b_a) \subseteq W$,

Choose $V = W$. For any $a \in [x,b)$ $x \le a < b$. $\Rightarrow \exists\ b_a$ s.t $x \le a < b_a < b$
$\Rightarrow [x,b_a] \in \mathcal{G}_a$ & $[x,b_a) \subseteq W$.

Hence proved.

1(b) Ans The topology generated by these sets

a set $A \subseteq \mathbb{R}$ is open if for every

$x \in A$. $\exists$ $[x, b_x) \in \mathcal{G}_x$ s.t

$[x, b_x) \subseteq A$.

$\Rightarrow \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} [x, b_x) \subseteq A$

$\bigcup_{x \in A} \{x\} = A$.

$\therefore \quad A = \bigcup_{x \in A} [x, b_x)$.

$\therefore$ A is ~~of~~ union of sets of the form.

$[x, b_x)$.

1(c). $\overline{\mathbb{Q}} = \{ a \in \mathbb{R} /$. every basic neighbourhood of a ~~or~~ has non empty intersection with $\mathbb{Q} \}$.

ie $x \in \mathbb{R}$ is ~~in~~ s.t $x \in \overline{\mathbb{Q}}$ if

$[x, b) \cap \mathbb{Q} \neq \phi \quad \forall \quad [x, b) \in \mathcal{G}_x$

ie $b \in \mathbb{R}$.

But by denseness of rationals we know that $[x, b)$ is

1(c) $\overline{\{1/n \mid n \in \mathbb{N}\}}$, Claim that the closure of $\{1/n \mid n \in \mathbb{N}\}$ is $\{1/n \mid n \in \mathbb{N}\} \cup \{0\}$.

Show that $0 \in \overline{\{1/n \mid n \in \mathbb{N}\}}$

and note if $x \neq 0$ & $x \notin \{1/n \mid n \in \mathbb{N}\}$,

then if $x < 0$ or $x > 1$, $\exists$ $[x, 0)$ ~~or~~ $[x, x+1)$ which have empty intersection with $\{1/n \mid n \in \mathbb{N}\}$

If $x \in (0,1)$. but $x \notin \{1/n \mid n \in \mathbb{N}\}$.

Then Consider

$\{ n \in \mathbb{N} \mid \frac{1}{n} > x \}$. This set is bounded above & hence has a largest element say $k \in \mathbb{N}$.

Then $[x, 1/k) \cap \{1/n \mid n \in \mathbb{N}\} = \emptyset$.

$\therefore$ $x \notin \overline{\{1/n \mid n \in \mathbb{N}\}}$. $\square$.

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1(c) Use similar arguments as before to show that $\{-1/n \mid n \in \mathbb{N}\} = \{-1/n \mid n \in \mathbb{N}\}$.

Pb(2). $\mathbb{R}^I = \{f : I \to \mathbb{R},\ f \text{ a func}^n\}$

(a) Denote $\mathcal{G}_f = \{U(f, F, \delta) \mid F \subseteq \mathbb{R} \text{ is finite}, \delta > 0\}$.

T.S.T $\{\mathcal{G}_f \mid f \in \mathbb{R}^I\}$ forms a neighbourhood base for some topology on $\mathbb{R}^I$.

(i) $f \in U(f, F, \delta)\ \forall\ U(f, F, \delta) \in \mathcal{G}_f$

since $|f(x) - f(x)| = 0 < \delta \quad \forall x \in \mathbb{R}$.

(ii) $g \in U(f, F_1, \delta_1) \cap U(f, F_2, \delta_2)$

$\Rightarrow \quad |g(x) - f(x)| < \delta_1 \quad \forall x \in F_1$

$|g(x) - f(x)| < \delta_2 \quad \forall x \in F_2$

Choose $\delta = \min\{\delta_1, \delta_2\}$ W.LOG let $\delta = \delta_1$

$\Rightarrow$. Then $g \in U(f, F_1 \cup F_2, \delta)$

$\Rightarrow |g(x) - f(x)| < \delta_1 < \delta_2 \quad \forall x \in F_1 \cup F_2$

$\therefore U(f, F_1 \cup F_2, \delta) \subseteq U(f, F_1, \delta_1) \cap U(f, F_2, \delta_2)$.

(iii). Let $W = U(f, F, \delta)$. Let $V = U(f, F, \delta/3)$.

For any $g \in V$. $|g(x) - f(x)| < \delta/3 \ \forall x \in F$.

Now Claim $U(g, F, \delta/3) \subseteq W$.

Reason. $h \in U(g, F, \delta/3)$

$$\Rightarrow |h(x) - g(x)| < \delta/3$$

$$\Rightarrow |h(x) - f(x)| \leq |h(x) - g(x)| + |g(x) - f(x)|$$
$$< \delta/3 + \delta/3 = 2\delta/3 < \delta.$$

$$\Rightarrow h \in U(f, F, \delta).$$

$\therefore \forall g \in V \ \exists \ U(g, F, \delta/3) \in \mathcal{Y}_g$ s.t.

$U(g, F, \delta/3) \subseteq W$.

$\therefore \{\mathcal{G}_f\}_{f \in \mathbb{R}^I}$ satisfies all the properties required for it to generate a top. for which it will be a neighbourhood base.

P.T.O.

2(b) T.ST $\{f\}$ is closed.

Let. $g \neq f$, & $g \in \mathbb{R}^I$.

T.ST: $\exists$ a basic nbd of g s.t. it has empty intersection with $\{f\}$.

<u>Proof</u>: If. $g \neq f$ then $\exists$ $x \in I$ s.t

$$g(x) \neq f(x).$$

$$\Rightarrow |g(x) - f(x)| = \varepsilon \neq 0.$$

Choose $F = \{x\}$ & $\delta = \varepsilon/2$,

Then. Show that $f \notin U(g, F, \delta)$

$$\Rightarrow g \notin \overline{\{f\}}.$$

$\Rightarrow \overline{\{f\}} = \{f\}$.

(3).(a) Done in class

(b) Show that $\mathcal{U}_{1Y}$ is finer than $\mathcal{U}_{2Y}$.

Question: If $\mathcal{U}_1$ is strictly finer that $\mathcal{U}_2$ is $\mathcal{U}_{1Y}$ strictly finer that $\mathcal{U}_{2Y}$?

Problems 4 & was discussed in class.

(5): The proof is similar to showing any separable metric space is second countable.

(6) (a). Show that for any metric space (X,d)
$\mathcal{G}_a = \{ B_d(a, 1/n) \mid n \in \mathbb{N} \}$, $a \in X$
is a neighbourhood base.

(b). Let X be a second countable topological space. i.e $\exists$ a base $\mathcal{B}$ for the topology on X. s.t $\mathcal{B}$ is countable.
Define $\mathcal{G}_a = \{ B \in \mathcal{B} \mid a \in B \}$ then $\mathcal{G}_a$ is countable as $\mathcal{G}_a \subseteq \mathcal{B}$.
Show that $\{\mathcal{G}_a\}$ is a neighbourhood base for X. ie for any $a \in X$, ? neighbourhood W of a. show that $\exists$ $B \in \mathcal{G}_a$ s.t $a \in B \subseteq W$.

P.T.O

9(c) show that $\mathbb{R}_\ell$ is separable.

Show that $\mathbb{Q}$ is a dense subset of $\mathbb{R}_\ell$.

Now show that $\mathbb{R}_\ell$ is separable. $\Rightarrow \mathbb{R}_\ell$ is

cond countable. Then by 6(a). $\mathbb{R}_\ell$

fir - countable.

7(a) Let $\mathbb{R}_<$ denote $\mathbb{R}$ with order topology

Since $\mathbb{R}_<$ has no least or largest

element. base for $\mathbb{R}_<$ is given by

$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}\}$

where $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

But this is the base for $\mathbb{R}$ with

usual topology. so. $\mathbb{R}_<$ is same as $\mathbb{R}$.

7(b). $I \times I = [0,1] \times [0,1]$ with lexicographic

order has least element $(0,0)$ &

largest element $(1,1)$.

The topology on $I \times I$ has a base.

consisting of the following sets

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$$((a,b),(c,d)) := \{(x,y) \in I \times I \mid a < x < c\}$$
$$\cup \{(x,y) \in I \times I \mid x = a,\ y > b.\}$$
$$\cup \{(x,y) \in I \times I \mid x = c,\ y < d\}.$$

if $a < c$.

$$\text{~~Sen~~ } ((a,b),(a,d)) = \{(a,y) \in I \times I \mid b < y < d\}$$

$$a > 0,\ [(0,0),(a,b)) = \{(x,y) \in I \times I \mid 0 < x < a\}$$
$$\cup \{(0,y) \in I \times I \mid 0 < y < b\}.$$

$$a < 1,\ ((a,b),(1,1)] = \{(x,y) \in I \times I \mid a < x < 1\}$$
$$\cup \{(a,y) \in I \times I \mid b < y < 1\}.$$

& there are sets of the form.

$[(0,0),(1,1)]$, $[(0,0),(0,b))$ and

$((1,b),(1,1)]$.

Now look at the pts of the form $(x,0)$ & $(1,y)$ in $I \times I$ and find all basic neighbourhoods around them.

7(c) Take $X = \mathbb{R}$ & $Y = \mathbb{N}$. Show that $(Y, \mathcal{C})$ is not a discrete space.

Y with order topology.