PARTIAL SOLUTIONS. TO TUTORIAL 4.

(1). False Show that $\mathbb{Q}$ as a subspace of $\mathbb{R}$ will not have discrete top Show that If $r, s \in \mathbb{Q}$ then $(r,s) \cap \mathbb{Q}$ is not closed in $\mathbb{Q}$. So there are subsets of $\mathbb{Q}$ which are not closed.

(2). Easy exercise. to show that $Y \subseteq (X,d)$ as a subspace is same as $(Y, d|_Y)$.

(3). Show that if $h : X \to \mathbb{R}$ is a continuous function then $\{x \in X / . h(x) = 0\}$ is a closed set

Then show that if f & g are cts then $f - g : X \to \mathbb{R}$ is also cts.

Note that for any $\varepsilon > 0$. & $a \in X$.

$\exists$ a nbd. V_1 of a s.t. & nbd V_2 of a.

$$V_1 \subseteq f^{-1}(.B(f(a), \varepsilon/_2))$$

$$\& \quad V_2 \subseteq g^{-1}(B(g(a), \varepsilon/_2))$$

Let. $W = V_1 \cap W_2$ then $x \in V_1 \cap V_2$

$\Rightarrow |f(x) - f(a)| < \varepsilon/_2$ & $|g(x) - g(a)| < \varepsilon/_2$

Therefore $\forall x \in V_1 \cap V_2$

$$|f(x) - g(x) - (f(a) - g(a))|$$
$$\leq |f(x) - f(a)| + |g(x) - g(a)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow a \in V_1 \cap V_2 \subseteq (f-g)^{-1} B(f-g(a), \varepsilon)$.

$\therefore f-$ is cts at a.

Now to show that if $D \subseteq \{x \in X \,/\, f(x) = g(x)\}$ where D is dense in X. then $f = g$.

$\bar{D} \subseteq \{x \in X \,/\, f(x) = g(x)\} \longrightarrow$ since this is closed

$\Rightarrow X \subseteq \{x \in X \,/\, f(x) = g(x)\} \subseteq X$

$\Rightarrow f = g$.

P.T.O

(4). Let $f : X \to Y$ be a homeomorphism

& (a) X be metrizable.

i.e. X has topology given by a metric

d.

Since f is a bijection we have.

$f^{-1}d : Y \times Y \to \mathbb{R}$,

is a metric.

Verify this

Now use the fact that f is a homeomorphism

& $f^{-1}d$ is a metric to show that

the metric top given by $(Y, f^{-1}d)$ is same

as the topology on Y.

4.
(b) If X has ctbly many open sets &.

f is a homeomorphism.

V in Y is open iff $f^{-1}(V)$ is open X.

Show that
$\{V \subseteq Y \mid V \text{ open}\} \longrightarrow \{f^{-1}(V) \mid f^{-1}(V) \text{ open in } X\}$

is a bijection

& hence Y has ctbly many open sets.

4(c) Similar to 4 (b).

(5) T.ST $(A \times B)^{\circ} = A^{\circ} \times B^{\circ}$.

A° is open in X & B° is open in Y

$\Rightarrow$ $A^{\circ} \times B^{\circ}$ is open in $X \times Y$

& $A^{\circ} \times B^{\circ} \subseteq A \times B$

$\Rightarrow$ $A^{\circ} \times B^{\circ} \subseteq (A \times B)^{\circ} \leftarrow$ largest open set in $A \times B$.

Now. let. $(x,y) \in (A \times B)^{\circ}$.

Then $A \times B$ is a nbd of (x,y)

i.e $\exists$ a open set. W s.t

$(x,y) \in W \subseteq A \times B$

But. $A \times B$ has base. $\{U \times V \mid U$ open in X & V open in Y

$\Rightarrow$ $\exists$ $U \times V$ s.t

$(x,y) \in U \times V \subseteq A \times B$

$\Rightarrow$. $x \in U \subseteq A$ & $y \in V \subseteq B$

$\Rightarrow$ $x \in A^{\circ}$ & $y \in B^{\circ}$.

$\Rightarrow$ $(x,y) \in A^{\circ} \times B^{\circ}$,

(6). Similarly ideas as (5).