Assignment 1 MA 5102
Autumn 2017, IIT Bombay

(1) Let $\mathrm{D}^n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$. Define the relation $x \sim y$ if $x, y \in \mathrm{S}^1$ on D^2. Show that the quotient space $\mathrm{D}^2/\mathrm{S}^1$ is homeomorphic to S^2. Similarly give a outline of proof for the fact that $\mathrm{D}^n/\mathrm{S}^{n-1}$ is homeomorphic to the space S^n.

(2) Describe the following quotient spaces.
   (a) The cylinder with each of its boundary circles identified to a point. (not the same point)
   (b) S^2 with its equator identified to a point.

(3) Let $f : X \to Y$ be a quotient map. Define a relation on X as $a \sim b$ if $f(a) = f(b)$.
   (a) Show that $\sim$ defines an equivalence relation on X.
   (b) Show that $X/\sim$ with quotient topology is homeomorphic to Y.

(4) Let X be a topological space . Show that $f : \mathrm{S}^1 \to X$ is homotopic to constant map if and only the map from f can be extended to a map $g : \mathrm{D}^2 \to X$ such that $g_{|\mathrm{S}^1} = f$.

(5) A subspace $i : A \to X$ (where i is the inclusion map) is said to be a deformation retract if there exists a *retract* map $r : X \to A$ such that $ri = \mathrm{id}_A$ and $ir \sim \mathrm{id}_X$. This is said to be a strong deformation retract if the homotopy $H : X \times I \to X$ between ir and id_X has the property that $H(a, t) = a$ for all $a \in A$ and $t \in I$.

   Show that there is a circle in the Mobius strip which is a strong deformation retract of the Mobius strip. Deduce that the Mobius strip and the cylinder are homotopy equivalent.

(6) Show that if X is connected and Y is homotopy equivalent to X then Y is connected.

(7) Prove that $r : \mathrm{D}^n \to \mathrm{S}^{n-1}$ is a retract if and only if S^{n-1} is contractible.

(8) Show that there is a circle in $\mathbb{R}^2 - \{(0, 0)\}$ which is a strong deformation retract of $\mathbb{R}^2 - \{(0, 0)\}$.