MA 581 ELEMENTS OF DIFFERENTIAL TOPOLOGY
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1. LECTURES WEEK 1 [1]

What is this course about? Even though the idea of differentiation and integration are defined for subsets of euclidean spaces $\mathbb{R}^n$ for a more general category of spaces. In this course we enlarge the category of objects for which differentiation is defined and explore the properties of this category of differentiable manifolds.

1.1. Recall: Euclidean Spaces and Smooth maps, Diffeomorphisms.

Before we generalize, let us recall a few basic definitions. We say that a map from $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $\underline{x} \in \mathbb{R}^n$ if there exists a linear transformation $L(\underline{x}) : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\underline{h}\to 0} \left| \frac{f(\underline{x}+\underline{h}) - f(\underline{x})}{\underline{h}} - L(\underline{x}) \right| = 0.$$

If we write $f(\underline{x}) = (f_1(\underline{x}), \ldots, f_m(\underline{x}))$ then $L(\underline{x})$ if it exists, is given by the linear transformation,

$$\begin{bmatrix} \dfrac{\partial f_1(x)}{\partial x_1} & \cdots & \dfrac{\partial f_1(x)}{\partial x_n} \\ \dfrac{\partial f_2(x)}{\partial x_1} & \cdots & \dfrac{\partial f_2(x)}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial f_m(x)}{\partial x_1} & \cdots & \dfrac{\partial f_m(x)}{\partial x_n} \end{bmatrix}.$$

Note, if all the partial derivatives of f_i's exist for all i and are continuous in a open set around $\underline{x}$, that is, if f is continuously differentiable at $\underline{x}$ then $Df(x)$ exists.

The function f is said to be $\mathcal{C}^n$ if it is n-times continuously differentiable, that is, if all the n-order partials of f_i's exists for all i. If f is $\mathcal{C}^n$ for all n then it is said to be $\mathcal{C}^\infty$ or smooth.

EXAMPLE:

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(2) Let $g : \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\} \to \mathbb{R}^3$ be defined as

$$g(x,y) = (x, y, \sqrt{1-x^2-y^2}).$$

REMARKS:

- Derivative of a linear transformation L is L.
- If f is differentiable then Df is a function from $U \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.
- Partial derivatives existing does not ensure that f is differentiable.

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- Chain rule holds. If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : V \subseteq \mathbb{R}^m \to \mathbb{R}^k$ are composable functions differentiable at $\underline{a}$ and $f(\underline{a})$ respectively then, $D(gf)(\underline{a}) = Dg(f(\underline{a}))D(f)(\underline{a})$.
- A smooth map $f : U \to V$ is said to be a diffeomorphism if f^{-1} exists and is a smooth map.

1.2. **(Differentiable) Manifolds.** Very often the subset U of $\mathbb{R}^n$ has more structure. For instance, an ant walking on a Mobius strip in $\mathbb{R}^3$ is aware of only two dimensions in which it can move (unless of course it could fly!) Very often we want to characterize this nature of subsets of Euclidean spaces ($\mathbb{R}^n$). For the purposes of this class we will assume the topological spaces we have are all Hausdorff.

Definition 1.1. A topological space X is said to be a differentiable manifold if it has an open cover $\{U_i\}_{i\in I}$ such that,

- there exist homeomorphisms (coordinate chart) $\phi_i : U_i \to \mathbb{R}^n$ for every i and,
- $\phi_i\phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a diffeomorphism if $U_i \cap U_j \neq \emptyset$.

The manifold X is said to have a dimension n. The maps ϕ_i^{-1} are called parametrizations. The collection (U_i, ϕ) is called an atlas on X.

If the maps in atlas are $\mathcal{C}^p$-isomorphisms then the manifold is called a $\mathcal{C}^p$ manifold.

Remark:

- If X is a subspace of $\mathbb{R}^k$ for some k, then we could alternately insist that the coordinate charts are diffeomorphisms, in which case the second condition is automatically satisfied. For purposes of this class we will often assume that $X \subset \mathbb{R}^n$. The above definition is often called an abstract differentiable manifold.
- An atlas gives a *differentiable structure* on the given topological space. Two different atlases give the same differentiable structure if they are compatible. That is, (U_i, ϕ_i) and (V_j, α_j) are compatible atlases on a space X if for nonempty $U_i \cap V_j$ $\phi_i\alpha_j^{-1}$ is $\mathcal{C}^p$-isomorphisms for all i and j.
- A topological manifold or (a PL-manifold) can be defined similarly by insisting that $\phi_i\phi_j^{-1}$ be a homeomorphism (or a peicewise linear map)

Examples

(1) The standard differentiable structure on $\mathbb{R}^n$ is given by the maximal altas containing the chart $(\mathbb{R}^n, i : \mathbb{R}^n \to \mathbb{R}^n)$.
(2) $\mathrm{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a one dimensional manifold. We know that locally the coordinate y can be determined by x via the function $\pm\sqrt{1 - x^2}$.

    Let $U_0 = \{(x, y) \in \mathrm{S}^1 \mid y > 0\}$, $U_1 = \{(x, y) \in \mathrm{S}^1 \mid y < 0\}$, $U_2 = \{(x, y) \in \mathrm{S}^1 \mid x < 0\}$ and $U_3 = \{(x, y) \in \mathrm{S}^1 \mid x > 0\}$. This covers S^1 and we have a homeomorphism from $f_i : U_i \to \mathbb{R}$ given by $f_i(x, y) = x$ if $i = 0, 1$ and $f_i(x, y) = y$ if $i = 2, 3$. This defines a coordinate chart on S^1. Note since $\mathrm{S}^1 \subseteq \mathbb{R}^2$ it is sufficient to verify f_i's are diffeomorphisms.
(3) Show that Torus is a 2 dimensional manifold.

(4) $\mathbb{C}^n$ is a $2n$-dimensional real manifold. In general the any k dimensional vector space V is a topological space once we fix a basis. More precisely the basis elements define a linear isomorphism $\phi : V \to \mathbb{R}^n$ and hence a topology on V. The same isomorphism then gives a differentiable structure on V since it will be a homeomorphism.
(5) Every open subset of $\mathbb{R}^n$ is a manifold. As a consequence $GL_n(\mathbb{R})$ is a manifold.
(6) Define $F(n,k) \subset \mathbb{R}^n \times \mathbb{R}^k$ to the space of sets of k-linearly independent vectors in $\mathbb{R}^n$. This is a manifold by noting that $F(n,k)$ is the set of all $n \times k$ matrices of rank k which is open in $\mathbb{R}^{nk}$.
(7) Note that $\mathbb{R}$ has other atlases which is not compatible with the standard atlas. For instance the homeomorphism $\mathbb{R} \to \mathbb{R}$ given by $x \to x^3$ is not compatible with the standard differentiable structure. However, these two structures will give *diffeomorphic* differentiable manifolds. Some of these possibilities can be answered by the classifying problem for n-dimensional manifolds. We will not discuss this in this course or the question as to how many different differentiable structures can a topological space have up to diffeomorphism.

Proposition 1.2. *The cartesian product of manifolds is a manifold.*

It is easy to verify that if (U_i, ϕ_i) and (V_j, α_j) are $\mathcal{C}^p$ atlases on M and N respectively then $(U_i \times V_j, \phi_i \times \alpha_j)$ define an $\mathcal{C}^p$-atlas on $M \times N$.

This gives further examples of manifolds and proves that a torus is a manifold.

We can get further examples by consider quotient spaces of subsets of Euclidean spaces. Of course not every quotient space of a manifold will give a manifold since quotient spaces aren't necessarily Hausdorff as you will see in your tutorial.

Recall that Y is said to be a quotient space of a topological space if $f : X \to Y$ is a onto map and $U \subseteq Y$ is an open subset of Y if and only if $f^{-1}(U)$ is open in X. Y is said to have the quotient topology. Quotient spaces can be thought of as the set of equivalence classes of a an equivalence relation $\sim$ on some topological space X, with quotient topology.

(1) Show that Mobius Strip, $M = I^2/\sim$ where $(0,y) \sim (0, 1-y)$ is a two dimensional manifold.
(2) Let $\mathrm{P}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z) \neq (0,0,0)\}/\sim$ where $(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ for all $0 \neq \lambda \in \mathbb{R}$. Then P^2 is a 2 dimensional manifold. Use an alternate description of P^2 as a quotient space of of S^2
(3) Define the Grassmanian space $G(n,k)$ to be the set of k-subspaces of $\mathbb{R}^n$. This is a topological space with as a quotient space of $F(n,k)$ with the relation $A \sim B$ if there exists a matrix $C \in GL(k)$ such that $A = CB$. Then $G(n,k)$ is a $n(n-k)$ manifold.