Tutorial III, MA 419
IIT Bombay, Autumn 2016

(1) How many onto homomorphism are there from $\mathbb{Z}_{20} \to \mathbb{Z}_8$? How many homomorphisms are there from $\mathbb{Z}_{20} \to \mathbb{Z}_8$?

(2) What are all the homomorphisms from $\mathbb{Z} \to \mathbb{Z}_4$? What are all the group homomorphisms from $\mathbb{Z}_n \to \mathbb{Z}$ for any $n \in \mathbb{N}$?

(3) Are the following groups cyclic? Why or why not? If the group is finite cyclic, find all its generators.
  (a) $(\mathbb{Q}^*, .)$,
  (b) $(\mathbb{Z}_{10}^{\times}, .)$.
  (c) $(\mathbb{Z}_{14}^{\times}, .)$.
  (d) $(\mathbb{Z}_{pq}^{\times}, .)$, where p and q are prime.
  (e) G is a group with $(p-1)$ elements of order p, p is a prime.
  (f) G is a group such that all its proper subgroups are cyclic.

(4) In general, if G is a finite cyclic group of order n. How many generators will it have? What if G is infinite?

(5) Give examples of matrix subgroups which are cyclic.

(6) Prove that an abelian group with two subgroups of order 2 must have a subgroup of order 4.

(7) Let $G = \{f : \mathbb{R} \to \mathbb{R}^* \mid f \text{ is a function}\}$. Show G is a group under multiplication. Is $H = \{f : R \to \mathbb{R}^* \mid f(1) = 1\}$ a subgroup?

(8) Let x belong to a group. If $x^2 \neq e$ and $x^6 = e$, prove that $x^4 \neq e$ and $x^5 \neq e$. What can be said about the order of x?

(9) Show that every group of order 3 is cyclic.

(10) Show that every group of order 4 is abelian. How many group of order 4 exist up to isomorphism?

(11) Let $(G, .)$ and $(K, .)$ be groups. Define operation $\star$ on $G \times K$ as follows $(g_1, k_1) \star (g_2, k_2) := (g_1.g_2, k_1.k_2)$.
  (a) Show that $(G \times K, \star)$ is a group.
  (b) Let $a \in G$ have order m and $x \in K$ have order n. What can you say about the order of (a, x) in $G \times K$?

(12) Let G be an abelian group. Prove that $\{g \in G \mid o(g) < \infty\}$ is a subgroup of G. This subgroup is called the torsion subgroup of G. For a fixed integer $n > 1$, find the torsion subgroup of $\mathbb{Z} \times \mathbb{Z}_n$.