Tutorial VII MA 419
Group Actions, N/C theorem*

(1) Verify if the following define group actions. If they are group actions, describe their orbits and stablizers subgroups.
   (a) $\mathbb{Z}_2 \times S_n \to S_n$, defined as follows for all $\sigma \in S_n$,
$$\begin{aligned} 0.\sigma &= \sigma \\ 1.\sigma &= (12) \circ \sigma \end{aligned}$$
   (b) $A_n \times S_n \to S_n$, defined as follows for all $\sigma \in S_n$ and $\alpha \in A_n$,
$$\alpha.\sigma := \alpha \circ \sigma.$$
   (c) $Z_4 \times D_{2.4} \to D_{2.4}$ defined as follows for all $x \in D_{2.4}$ and $n \in \mathbb{Z}_4$,
$$n.x = x^n.$$
   (d) Let G be a group of order p, where p is a prime. Define $\mathbb{Z}_p \times G \to G$ for all $n \in \mathbb{Z}_p$ and $x \in G$, as follows ;
$$n.x = x^n.$$

(2) If $O(G) = pn$ with $p > n$, p prime, and H is a subgroup of order p, then H is normal in G. (Hint: Let G act on the left cosets of H, then $\mathrm{Ker}\phi < H$, where ϕ is the morphism induced by the action.)

(3) Let G act on a set S. Let s, t be elements of S, and let $x \in G$ be such that $xs = t$. Show that $G_t = xG_sx^{-1}$.

(4) Let G operate on a set A and let $\phi : G \to S_A$ be the corresponding group homomorphism. Prove that $\mathrm{Ker}\phi = \cap_{a \in A} G_a$.

(5) Let N be a normal subgroup of order 5 in a group G of odd order. Show that N is contained in the center of G. (Apply N/C theorem from tutorial 6)

(6) Show that every group of order 15 is cyclic.

(7) Show that any abelian group of order pq where $p \neq q$ are primes is cyclic.

(8) Find all conjugacy classes and their sizes in the following groups: D_8, Q_8 and A_4.

(9) Show that the center of S_n is identity for all $n \geq 3$.

(10) Prove that under any automorphism fo D_8 r has at most two possible images and s has at most 4 possible images. Deduce that $|\mathrm{Aut}(D_8)| \leq 8$.

(11) Use the fact that D_8 is normal in D_{16} to prove that $\mathrm{Aut}(D_8) \cong D_8$.

(12) Prove that a group of order 1225 is abelian.

(13) Let G be a group of order 3825 . Prove that if H is a normal subgroup of order 17 in G then $H \leq Z(G)$.

(14) Prove that any group of order 108 must have either a normal subgroup of order 27 or a normal subgroup of order 9.

(15) How many 3-Sylow subgroups and 5-Sylow subgroups does S_5 have?

(16) Classify groups of order 28. Show that up to isomorphism there are two abelian groups of order 28 and two non-abelian.

(17) Classify groups of order 30 using Sylow's theorem.

(18) Let G be a group of order 60, let P be a Sylow 5-subgroup of G and let Q be a Sylow 3-subgroup of G.
   (a) Prove that if P is not normal in G then G is simple.
   (b) Use the previous part to show that A_5 is simple. (Hint: Show that A_5 has more than one 5-Sylow subgroup.)
   (c) If G is a simple group of order 60 then G is isomorphic to a subgroup of A_6.
   (d) If G is a simple group of order 60 then G is isomorphic to A_5.