# Tutorial IX MA 419
# Rings, Integral Domains

(1) Show that $\mathbb{Z}_3[i]$ is an integral domain. In general, is $\mathbb{Z}_p[i]$ an integral domain for $p$ prime?

(2) Give examples of following rings:
  (a) A finite ring without identity.
  (b) A finite non-commutative ring.
  (c) A commutative ring without identity.
  (d) A ring with elements $a$ and $b$ such that $ab = 0$ but $ba \neq 0$.
  (e) A non-commutative ring with exactly 16 elements.

  (f) A ring with two elements $a$ and $b$ such that

(3) Show that if $R$ is a ring such that $(R, +)$ is a cyclic group then $R$ is commutative.

(4) Define $ma = a + \cdots + a$ ($m$-times) . Then show that
  (a) $(ma)(nb) = (mn)(ab)$.
  (b) $n(-a) = -(na)$.
  (c) $m(ab) = (ma)b = a(mb)$.

(5) Which of the following are integral domains? Which ones are fields?
  (a) $\mathbb{Z}_n[i]$.
  (b) $\mathbb{Q}[i]$.
  (c) $\mathbb{Z}[\sqrt{2}]$.
  (d) $\mathbb{Z}[x]$.
  (e) $\mathbb{R}[x]$.
  (f) $M_2[\mathbb{R}]$.
  (g) $M_2[\mathbb{Z}_3]$.

(6) Let $R$ be a ring. An element $a \in R$ is said to be nilpotent if for some $n > 0$, $a^n = 0$. Show that if $R$ has identity then $1 - a$ is invertible.

(7) Let $A = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a function}\}$ be the ring of all real valued functions on $\mathbb{R}$ under addition and multiplication in $\mathbb{R}$. Find all the zero divisors and nilpotent elements in $A$.

(8) Describe all ring homomorphisms from the ring $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. In each case describe the kernel and image.

(9) Let $G = \{g_1, g_2, \times, g_n\}$ be a finite group. Let $R$ be a commutative ring with identity. Prove that the element $w = 1.g_1 + \ldots + 1.g_n$ is in the center of the group ring $RG$.

(10) Define the set $R[[x]]$ if formal power series in the indeterminate $x$ with coeffecients from $R$ to be all formal sums $\sum_{n=0}^{\infty} a_n x^n$. Define addition and multiplication in $R[[x]]$ as

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty} (a_n + b_n) x^n \\ \sum_{n=0}^{\infty} a_n x^n . \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}) x^n . \end{aligned}$$

Show that $R[[x]]$ is a ring.

(11) Prove that if $R$ is an integral domain then $R[[x]]$ is also an integral domain.

(12) Let $C[0,1]$ denote the ring of real-valued continuous functions on $[0,1]$. Then the function $\phi : C[0,1] \to \mathbb{R}$ defined as $\phi(f) = \int_0^1 f(t)dt$ is a ring homomorphism.

(13) Prove that the ring $M_2(\mathbb{R})$ contains a subring that is isomorphic to $\mathbb{C}$.

(14) Let $R$ be a commutative ring with identity. Show that if $R$ is an integral domain then $R[x]^\times = R^\times$.

(15) Let $R$ be a commutative ring with identity. Show that $p(x) = \sum_{i=0}^{n} a_n x^n \in R[x]$ is a unit if $a_0 \in R^\times$ and $a_i$ is nilpotent for all $i \geq 1$.