ASSIGNMENT 7

(1) How many elements of order 7 must there be in a simple group of order 168?

(2) Let G be a group of order pq, with $p < q$.
  (a) If p does not divide $q-1$, then G is cyclic.
  (b) If p divides $q-1$, then there exists a unique non-abelian group of order pq.

(3) Let G be a group of order 30.
  (a) Show that a 3-Sylow subgroup or a 5-Sylow subgroup is normal in G. (HINT: Count the number of elements which have order 3 or 5)
  (b) Use the previous part to show that the 3-Sylow and 5-Sylow subgroup are normal.
  (c) Show that G has a normal subgroup of order 15. Use the previous problem to show that this subgroup is cyclic. Let β be a generator for this subgroup.
  (d) Classify all groups of order 30. (HINT: By Cauchy's theorem there is an element of order 2, call it a. Consider $a\beta a^{-1}$, where β is as in the previous part.)

(4) Let R be a normal p-subgroup of G (**not necessarily a Sylow subgroup**).
  (a) Show that R is contained in every p-Sylow subgroup of G.
  (b) Suppose that S is another normal p-subgroup of G then RS is also a normal p-subgroup of G.
  (c) Show that the subgroup $O_p(G)$, defined as the subgroup generated by all the normal p subgroups of G, is the largest normal p subgroup of G. Show that $O_p(G)$ equals the intersection of all the Sylow p subgroups of G.
  (d) Prove that $\bar{G} = G/O_p(G)$ has no nontrivial normal p subgroups.

(5) Consider the $n \times n$ determinant polynomial (this is a polynomial in n^2 variables)

$$\det(X_{i,j}) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} X_{i,\sigma(i)}.$$

In the above, $sgn(\sigma) = 1$ if σ is a product of an even number of permutations and $sgn(\sigma) = -1$ if σ is a product of an odd number of permutations. The $X_{i,j}$ are the variables. For an $n \times n$ matrix A with entries in $\mathbb{C}$ (complex numbers) we define the determinant of A to be

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

That is, $\det(A)$ is the polynomial $\det(X_{i,j})$ evaluated at $X_{i,j} = a_{i,j}$. We want to show in this exercise that $\det(AB) = \det(A)\det(B)$.
  (a) Show that if A is a matrix in which two columns are equal, then $\det(A) = 0$. Similarly, if two rows are equal.
  (b) Let A be a matrix, and denote it by $[C_1|C_2 \ldots |C_n]$, where C_i is the i^{th} column of A. Let B be a matrix which looks like $[C_1|C_2 \ldots |C_i'| \ldots |C_n]$, that is, A and B differ only in the i^{th} column. Let C be the matrix which looks like $[C_1|C_2 \ldots |C_i + C_i'| \ldots |C_n]$. Show that $\det(C) = \det(A) + \det(B)$.
  (c) Let $H = \{B \in M_n(\mathbb{C}) \mid \det(AB) = \det(A)\det(B)\ \forall A \in M_n(\mathbb{C})\}$ Show that if $B_1, B_2 \in H$ then $B_1 B_2 \in H$. Show that Id is in H.

(d) Let $E_{i,j}$ denote the $n \times n$ matrix whose ij^{th} entry is 1 and all other entries are 0. For a complex number α and $i \neq j$, show that $Id + \alpha E_{i,j}$ is in H. (REMARK: This is the same as saying that adding a scalar multiple of a column to another column does not change the determinant)

(e) Show that $\text{diag}(1, 1, \ldots, 1, \alpha, 1, \ldots, 1)$ is in H. (REMARK: This is the same as saying that multiplying a column with a scalar changes the determinant by that scalar)

(f) For $\sigma \in S_n$ define a permutation matrix E_σ by $(E_\sigma)_{i,j} = \delta_{i,\sigma(j)}$. This matrix is obtained by permuting the columns of the identity matrix. Show that $\det(E_\sigma) = sgn(\sigma)$. Show that $\det(AE_\sigma) = \det(A)\det(E_\sigma)$. Thus, $E_\sigma \in H$.

(g) The above show that H contains elements of the type: $Id+\alpha E_{i,j}$ $(i \neq j)$, $\text{diag}(1, 1, \ldots, 1, \alpha, 1, \ldots, 1)$, E_σ $(\sigma \in S_n)$. Part (a) shows that H is closed under matrix multiplication. Show that every element of $M_n(\mathbb{C})$ can be written as a product of elements of H. This shows that $\det(AB) = \det(A)\det(B)$ for any two matrices A, B.

(h) Show that $\det : GL_n(\mathbb{C}) \to \mathbb{C}^*$ is a group homomorphism.

(6) Let G and H be groups such that there is a prime p such that $p|\#G$ and $p|\#H$. Show that $G \times H$ is not cyclic. (HINT: A cyclic group has a unique subgroup of order d if d divides the order of the group)

(7) Let G be a group. Let

$$\text{Aut}(G) := \{\phi : G \to G \mid \phi \text{ is a bijective group homomorphism}\}$$

Show that the above is a group under composition.

(a) What is $\#\text{Aut}(\mathbb{Z}/p^2\mathbb{Z})$? (HINT: Any group homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ is completely determined by the image of $\bar{1}$, why?)

(b) Let H be the group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let $e_1 := (\bar{1}, \bar{0})$ and $e_2 := (\bar{0}, \bar{1})$ show that there is a 1-1 correspondence betweeb group homomorphisms $\phi : H \to H$ and elements of $H \times H$. When does an element of $H \times H$ correspond to an isomorphism? Use this to compute $\#\text{Aut}(H)$.

(8) Let G be a group of order $315 = 9 \cdot 7 \cdot 5$. Assume that the 3-Sylow subgroup of G is normal, call it P. Show that G/P is cyclic. Use this to show that G contains an element β of order 35. Show that there is a group homomorphism $G \to \text{Aut}(P)$ given by $g \mapsto \phi_g$, where $\phi_g(x) = gxg^{-1}$. Use the previous exercise to show that $\beta \mapsto Id$. Use this to show that there is a group homomorphism $P \times \mathbb{Z}/35\mathbb{Z} \to G$ which is an isomorphism. Thus, G is an abelian group.