MA-207 Differential Equations II

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References

Elementary differential equations with boundary value problems
by William F. Trench (available online)

Differential Equations with Applications and Historical Notes
by George F. Simmons

Welcome to MA 207, a sequel to MA 108. We begin by reviewing elementary functions, which were discussed in MA 108.

A function $f : \mathbb{R} \to \mathbb{R}$ of the type

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \qquad a_i \in \mathbb{R}$$

is called a polynomial function.

Example

$x^3 + 2x + 5$

A rational function is a quotient of polynomial functions.

Example

$$\frac{x^3 + 3x + 2}{x^5 + 2x^3 + 5},$$

A function $y = f(x)$ is called algebraic if it satisfies an equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \ldots + P_1(x)y + P_0(x) = 0$$

for some n, where each $P_i(x)$ is a polynomial.

Next we have

1. trigonometric functions, for example, $\sin x, \ \cos x, \ \tan x$
2. inverse trigonometric functions, for example $\sin^{-1} x, \ \cos^{-1} x, \ \tan^{-1} x$
3. exponential functions, for example $e^x, \ \log x$

A elementary function is one which can be obtained by adding, subtracting, multiplying, dividing and composing any of the above functions.

Thus

$$y = tan\left[\frac{xe^{1/x^2} + \tan^{-1}(1+x^2) + \sqrt{x^2+3}}{\sin x \cos 2x - \sqrt{\log x} + x^{3/2}}\right]^{1/3}$$

is an elementary function.

Beyond elementary functions lie the special functions, for example, Gamma function, Beta function, Riemann zeta function etc.

Definition

The Riemann zeta function is defined on the set $\{s \in \mathbb{C} \,|\, \mathrm{Re(s)} > 1\}$ by

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$$

It is a non-trivial theorem that the zeta function extends to the whole plane as a meromorphic function. (Explaining this term is beyond the scope of this course)

The Riemann hypothesis states that all the nontrivial zeros of the zeta function lie on the line $\mathrm{Re}(s) = \frac{1}{2}$.

This is one of the millennium problems and has a prize of 1 million US dollars.

Large number of special functions arise as solutions of 2nd order linear ODE. Suppose we want to solve

$$y'' + y = 0$$

Then elementary functions $y = \sin x$ and $y = \cos x$ are solutions.

Suppose we want to solve

$$xy'' + y' + xy = 0$$

This equation can not be solved in terms of elementary functions.

Let $y_1(x)$ be one solution of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x), q(x)$ continuous. Then we can try to use the method of variation of parameters to find another linearly independent solution, that is, put

$$y_2 = u(x)y_1(x)$$

in the ODE and solve for $u(x)$.

Question. How to find the 1st solution?

For this, we will solve our ODE in terms of power series.

Let us review power series, which is used throughout in this course.

Definition (Power series)

For real numbers $x_0, a_0, a_1, a_2, \ldots$, an infinite series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n := a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots .$$

is called a power series in $x - x_0$ with center x_0.

For a real number x_1, if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x_1 - x_0)^n$$

exists and is finite, then we say the power series converges at the point $x = x_1$. In this case, the sum of the series is the value of the limit.

If the series does not converge at x_1, that is, either limit does not exist or it is $\pm\infty$, then we say the power series diverges at x_1.

Theorem

For any power series,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

exactly one of these statements is true.

1. *The power series converges only for* $x = x_0$.
2. *The power series converges for all values of* x.
3. *There is a positive number* $0 < R < \infty$ *such that the power series converges if* $|x - x_0| < R$ *and diverges if* $|x - x_0| > R$.

R is called the radius of convergence of the power series.

We define $R = 0$ in case (i)
and $R = \infty$ in case (ii).

Question. How to compute the radius of convergence?

Theorem

- *(Ratio test) If $a_n \neq 0$ for all n and*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

- *(Root test)* $\limsup_{n\to\infty} |a_n|^{1/n} = L$

Then radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is* $\boxed{R = 1/L}$.

For $L = 0$, we get $R = \infty$ and for $L = \infty$, we get $R = 0$.

Theorem

Let $R > 0$ *be the radius of convergence of the power series*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$.

For $R = \infty$, we write $(x_0 - R, x_0 + R) = (-\infty, \infty) = \mathbb{R}$.

The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.

Example

Find the radius of convergence and interval of convergence (if $R > 0$) of the following three series

$$(i)\ \sum_{0}^{\infty} n!x^n \quad (ii)\ \sum_{10}^{\infty} (-1)^n \frac{x^n}{n^n} \quad (iii)\ \sum_{0}^{\infty} 2^n n^3 (x-1)^n$$

$$(i)\ \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty} (n+1) = \infty$$

So $R = 0$ in case (i).

Similarly, in case (ii) $R = \infty$ and in case (iii) $R = 1/2$.

Interval of convergence : in case (ii) $(-\infty, \infty)$ and in case (iii) $(1/2, 3/2)$

Theorem

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$. We assume $\boxed{R > 0}$

- *We can define a function $f : (x_0 - R, x_0 + R) \to \mathbb{R}$ by*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- *f is infinitely differentiable* $\forall\ x \in (x_0 - R, x_0 + R)$.

- *The successive derivatives of f can be computed by differentiating the power series term-by-term, that is*

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \qquad \ldots$$

$$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\ldots(n-k+1)\, a_n (x - x_0)^{n-k}$$

Theorem (continued ...)

- *The power series representing the derivatives $f^{(n)}(x)$ have same radius of convergence R.*
- *We can determine the coefficients a_n (in terms of derivatives of f at x_0) as*

$$f(x_0) = a_0, \quad f'(x_0) = a_1, \quad f''(x_0) = 2a_2, \ldots$$

In general,

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

- *We can also integrate the function $f(x) = \sum_0^\infty a_n (x - x_0)^n$ term-wise that is if $[a, b] \subset (x_0 - R, x_0 + R)$, then*

$$\int_a^b f(x)\, dx = \sum_{n=0}^\infty a_n \int_a^b (x - x_0)^n\, dx = \sum_0^\infty \frac{a_n}{n+1} (x - x_0)^{n+1}$$

Example (Power series representation of elementary functions)

(i) $$e^x = \sum_0^\infty \frac{x^n}{n!} \qquad -\infty < x < \infty$$

(ii) $$\sin x = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad -\infty < x < \infty$$

(iii) $$\frac{1}{1-x} = \sum_0^\infty x^n \qquad -1 < x < 1$$

(iv) $$\frac{d}{dx}(\sin x) = \sum_0^\infty (-1)^n \frac{d}{dx}\left(\frac{x^{2n+1}}{(2n+1)!}\right)$$

$$= \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

Theorem

(i) Power series representation of f in an open interval I containing x_0 is unique, that is, if

$$f(x)=\sum_0^\infty a_n(x-x_0)^n=\sum_0^\infty b_n(x-x_0)^n$$

for all $x\in I$, then $a_n=b_n\ \forall\ n$.
(ii) If

$$\sum_0^\infty a_n(x-x_0)^n=0$$

for all $x\in I$, then $a_n=0$ for all n.

Proof. *(i)*

$$a_n=\frac{f^{(n)}(x_0)}{n!}=b_n \qquad \text{for all} \quad n.$$

It is clear that (ii) follows from (i).

Algebraic operations on power series

Definition

If $$f(x) = \sum_{0}^{\infty} a_n (x - x_0)^n \qquad g(x) = \sum_{0}^{\infty} b_n (x - x_0)^n$$

have radius of convergence R_1 and R_2 respectively, then

$$c_1 f(x) + c_2 g(x) := \sum_{0}^{\infty} (c_1 a_n + c_2 b_n)(x - x_0)^n$$

has radius of convergence $\boxed{R \geq \min \{R_1, R_2\}}$ for $c_1, c_2 \in \mathbb{R}$.

Further, we can multiply the series as if they were polynomials, that is

$$f(x)g(x) = \sum_{0}^{\infty} c_n (x - x_0)^n; \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

It also has radius of convergence $R \geq \min \{R_1, R_2\}$.

Example

Find the power series expansion for $\cosh x$ in terms of powers of x^n.

$$\begin{aligned}
\cosh x &= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{1}{2}\left[1+(-1)^n\right]\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}
\end{aligned}$$

Since radius of convergence for Taylor series of e^x and e^{-x} are ∞, the power series expansion of $\cosh x$ is valid on $\mathbb{R}$.

Shifting the summation index

If $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \implies f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$

Let us rewrite the series for $f'(x)$ in powers of $(x - x_0)^n$. Put $r = n - 1$, we get

$$f'(x) = \sum_{r=0}^{\infty} (r+1) a_{r+1} (x - x_0)^r$$

Similarly,

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n (x - x_0)^{n-k} \\ &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\dots(n+1) a_{n+k} (x - x_0)^n \end{aligned}$$

In general, $\left[\sum_{n=n_0}^{\infty} b_n (x - x_0)^{n-k} = \sum_{n=n_0-k} b_{n+k} (x - x_0)^n \right]$

Example

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Write $(x-1)f''$ as a power series around 0.

$$\begin{aligned}
(x-1)f'' &= xf'' - f'' \\
&= x\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n \\
&= \sum_{n=0}^{\infty} \left[(n+1)n a_{n+1} - (n+2)(n+1)a_{n+2}\right] x^n
\end{aligned}$$

Example (Solving ODE)

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

for all x in an open interval I containing $x_0 = 1$.

- Find the power series of y' and y'' in terms of $x-1$ in the interval I. Use these to express the function

  $$(1+x)y'' + 2(x-1)^2 y' + 3y$$

  as a power series in $x-1$ on I.
- Find necessary and sufficient conditions on the coefficients a_n's, so that $y(x)$ is a solution of the ODE

  $$(1+x)y'' + 2(x-1)^2 y' + 3y = 0$$

Example (Continue ...)

Solution. Write the ODE in $(x-1)$, that is

$$(1+x)y'' + 2(x-1)^2y' + 3y = (x-1)y'' + 2y'' + 2(x-1)^2y' + 3y$$

Express each of $(x-1)y''$, $2y''$, $2(x-1)^2y'$ and $3y$ as a power series in powers of $(x-1)$ and add them.

$$\begin{aligned}(x-1)y'' &= (x-1)\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} \\ &= \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-1)^n \\ &= \sum_{n=0}^{\infty}(n+1)na_{n+1}(x-1)^n\end{aligned}$$

Example (Continue . . .)

$$2y'' = \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2}$$

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n$$

$$2(x-1)^2y' = 2(x-1)^2 \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

$$= \sum_{n=1}^{\infty} 2na_n(x-1)^{n+1}$$

$$= \sum_{n=2}^{\infty} 2(n-1)a_{n-1}(x-1)^n$$

$$= \sum_{n=0}^{\infty} 2(n-1)a_{n-1}(x-1)^n \quad (a_{-1} = 0)$$

Example (Continue ...)

We have

$$(x-1)y'' = \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n$$

$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n$$

$$2(x-1)^2y' = \sum_{n=0}^{\infty} 2(n-1)a_{n-1}(x-1)^n \quad (a_{-1}=0)$$

Now we get

$$(x-1)y'' + 2y'' + 2(x-1)^2y' + 3y = \sum_{n=0}^{\infty} b_n(x-1)^n$$

where

$$b_n = (n+1)na_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n$$

Example (Continue . . .)

$$y(x) = \sum_{0}^{\infty} a_n (x-1)^n$$

is the solution of the ODE

$$(x-1)y'' + 2y'' + 2(x-1)^2 y' + 3y = 0$$

on the open interval I containing 1 if and only if

$$\sum_{n=0}^{\infty} b_n (x-1)^n = 0 \text{ on } I \Longleftrightarrow b_n = 0 \quad \text{for all} \quad n$$

that is a_n's satisfy the following recursive relation

$$(n+1)na_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n = 0$$

for all n.

Definition

If a function $f(x)$ is infinitely differentiable at x_0, then the Taylor series of f at x_0 is defined as the power series

$$TS\, f|_{x_0} := \sum_{0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

When $x_0 = 0$, the series is also called the Maclaurin series of f.

Example

The function $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
is infinitely differentiable at 0. But $f^{(n)}(0) = 0$ for all n.

Hence the Taylor series of f at 0 is the constant function taking value 0.

Therefore Taylor series of f at 0 does not converge to function $f(x)$ on any open interval around 0.

Definition

Suppose

- $f(x)$ is infinitely differentiable at x_0; and
- Taylor series of f at x_0 converges to $f(x)$ for all x in some open interval around x_0;

Then f is called analytic at x_0.

Example

The function $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
is not analytic at 0. Here 2nd condition fails.
However, f is analytic at all $x \neq 0$.

Theorem (Analytic functions)

1. *If* $f(x)$ *and* $g(x)$ *are analytic at* x_0*, then* $f(x) \pm g(x)$ $f(x)g(x)$ $\quad f(x)/g(x)$ *(if* $g(x_0) \neq 0$*) are analytic at* x_0*.*
2. *If* $f(x)$ *is analytic at* x_0 *and* $g(x)$ *is analytic at* $f(x_0)$*, then* $g(f(x)) := (g \circ f)(x)$ *is analytic at* x_0*.*
3. *If a power series* $\displaystyle\sum_0^\infty a_n(x-x_0)^n$ *has radius of convergence* $R > 0$*, then the function* $f(x) := \displaystyle\sum_0^\infty a_n(x-x_0)^n$ *is analytic at all points* $x \in (x_0 - R, x_0 + R)$*.*

Example

The function $f(x) = x^2 + 1$ is analytic everywhere. Since $x^2 + 1$ is never 0, the function $h(x) := \frac{1}{x^2+1}$ is analytic everywhere.

However, there is no power series around 0 which represents $h(x)$ everywhere.

If there were such a power series, then by uniqueness, it has to be the power series expansion of $h(x)$ around 0, which is

$$1 - x^2 + x^4 - x^6 + \cdots$$

However, the radius of convergence of this is $R = 1$.

In fact, for any x_0, there is no power series around x_0 which represents $h(x)$ everywhere.

Theorem

Let

$$F(x) = \frac{N(x)}{D(x)} \quad \left(\textit{example} \;\; F(x) = \frac{x^3-1}{x^2+1}\right)$$

be a rational function, where $N(x)$ and $D(x)$ are polynomials without any common factors, that is they do not have any common (complex) zeros. Let $\alpha_1, \ldots, \alpha_r$ be distinct complex zeros of $D(x)$.

Then $F(x)$ is analytic at all x except at $x \in \{\alpha_1, \ldots, \alpha_r\}$.

If x_0 is different from $\{\alpha_1, \ldots, \alpha_r\}$, then the radius of convergence R of the Taylor series of F at x_0

$$TS\, F_{x_0} = \sum_{0}^{\infty} \frac{F^{(n)}(x_0)}{n!}(x-x_0)^n$$

is given by

$$R = \textit{min}\,\{|x_0-\alpha_1|, |x_0-\alpha_2|, \ldots, |x_0-\alpha_r|\}$$

Example

If

$$F(x) = \frac{N(x)}{D(x)} = \frac{(2+3x)}{(4+x)(9+x^2)}$$

then $D(x)$ has zeros at -4 and $\pm 3\iota$, where $\iota = \sqrt{-1}$.

Hence F is analytic at all x except at $x \in \{-4, \pm 3\iota\}$.

If $x = 2$, then radius of convergence of Taylor series of F at $x = 2$ is

$$\min\{|2+4|, |2+3\iota|, |2-3\iota|\} = \min\{6, \sqrt{13}\} = \sqrt{13}$$

If $x = -6$, then radius of convergence of Taylor series of F at $x = -6$ is

$$\min\{|-6+4|, |-6 \pm 3\iota|\} = \min\{2, \sqrt{45}\} = 2$$

Power series solution of ODE

Theorem (Existence Theorem)

If $p(x)$ and $q(x)$ are analytic functions at x_0, then every solution of

$$y'' + p(x)y' + q(x)y = 0$$

is also analytic at x_0; and therefore any solution can be expressed as

$$y(x) = \sum_{0}^{\infty} a_n (x - x_0)^n$$

If $R_1 =$ radius of convergence of Taylor series of $p(x)$ at x_0,

$R_2 =$ radius of convergence of Taylor series of $q(x)$ at x_0,

then radius of convergence of $y(x)$ is at least min$(R_1, R_2) > 0$.

In most applications, $p(x)$ and $q(x)$ are rational functions, that is quotient of polynomial functions.

Series solution of ODE

Example

Let us solve $\boxed{y'' + y = 0 \qquad (1)}$ by power series method.

Compare with $y'' + p(x)y' + q(x)y = 0$,
$p(x) = 0$ and $q(x) = 1$ are analytic at all x.
We can find power series solution in $x - x_0$ for any x_0.
Let us assume $x_0 = 0$ for simplicity.
By existence theorem, all solution of (1) can be found in the form

$$y(x) = \sum_{0}^{\infty} a_n x^n$$

and the series will have ∞ radius of convergence.
Since

$$y'' = \sum_{2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Example (Continue ...)

$$y'' + y = \sum_{0}^{\infty}((n+2)(n+1)a_{n+2} + a_n)x^n = 0$$

By uniqueness of power series in $x - x_0$ with positive radius of convergence, we get the recursion formula

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$\implies a_{n+2} = \frac{-1}{(n+2)(n+1)}a_n \quad \forall n$$

Therefore,

$$a_2 = \frac{-1}{2.1}a_0, \quad a_4 = \frac{-1}{4.3}a_2 = \frac{1}{4!}a_0 \ \ldots \ a_{2n} = (-1)^n\frac{1}{(2n)!}a_0$$

$$a_3 = \frac{-1}{3.2}a_1, \quad a_5 = \frac{-1}{5.4}a_3 = \frac{1}{5!}a_1 \ \ldots \ a_{2n+1} = (-1)^n\frac{1}{(2n+1)!}a_1$$

Example (Continue ...)

Define

$$y_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \qquad (a_0 = 1, a_1 = 0)$$

$$y_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \qquad (a_0 = 0, a_1 = 1)$$

Then

$$y(x) = \sum_{0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

is a general solution of the ODE (1).

In this case, $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Thus, $y(x)$ is an elementary function. In general, however, the solution may not be an elementary function.

We don't need to check the series for converges, since the existence theorem guarantees that the series converges for all x.

Steps for Series solution of linear ODE

1. Write ODE in standard form $y'' + p(x)y' + q(x)y = 0$.
2. Choose x_0 at which $p(x)$ and $q(x)$ are analytic. If boundary conditions at x_0 are given, choose the center of the power series as x_0.
3. Find minimum of radius of convergence of Taylor series of $p(x)$ and $q(x)$ at x_0.
4. Let $y(x) = \sum_0^{\infty} a_n (x - x_0)^n$, compute the power series for $y'(x)$ and $y''(x)$ at x_0 and substitute these into the ODE.
5. Set the coefficients of $(x - x_0)^n$ to zero and find recursion formula.
6. From the recursion formula, obtain (linearly independent) solutions $y_1(x)$ and $y_2(x)$. The general solution then looks like $y(x) = a_1 y_1(x) + a_2 y_2(x)$.

The following ODE's are classical:

- Bessel's equation :

  $$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

  It occurs in problems displaying cylindrical symmetry, example diffusion of light through a circular aperture, vibration of a circular head drum, etc.

- Airy's equation :

  $$y'' - xy = 0$$

  It occurs in astronomy and quantum physics.

- Legendre's equation :

  $$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

  It occurs in problems displaying spherical symmetry, particularly in electromagnetism.

In this course, we will consider ODE

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

with $P_i(x)$ polynomials for $i = 0, 1, 2$ without any common factor.

If we write ODE in the standard form

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

we see that if x_0 is not a zero of $P_0(x)$, then
$P_1(x)/P_0(x)$ and $P_2(x)/P_0(x)$ will be analytic at x_0
hence we can find the series solution of ODE in the form

$$y(x) = \sum_0^\infty a_n(x - x_0)^n$$

When x_0 is a zero of $P_0(x)$, then x_0 is called a singular point of ODE. This case will be considered later.

Example

Find the power series in x for the general solution of

$$(1 + 2x^2)y'' + 6xy' + 2y = 0$$

Solution. Note that 0 is not a zero of $P_0(x) = 1 + 2x^2$, hence the series solution in powers of x exists.

Put $y = \sum_0^\infty a_n x^n$ in the ODE, we get

$$\begin{aligned}
&(1 + 2x^2)y'' + 6xy' + 2y \\
&= y'' + 2x^2 y'' + 6xy' + 2y \\
&= \sum_0^\infty ((n+2)(n+1)a_{n+2} + 2n(n-1)a_n + 6na_n + 2a_n)x^n
\end{aligned}$$

$$\implies (n+2)(n+1)a_{n+2} + [2n(n-1) + 6n + 2]a_n = 0$$

Example (Continue ...)

$$\implies a_{n+2} = -\frac{2n^2+4n+2}{(n+2)(n+1)}\, a_n = -2\frac{n+1}{(n+2)}\, a_n \quad n \geq 0$$

Since indices on left and right differ by 2, we write separately for $n = 2m$ and $n = 2m+1, \; m \geq 0$, so

$$a_{2m+2} = -2\frac{2m+1}{2m+2}\, a_{2m} = -\frac{2m+1}{m+1}\, a_{2m}$$

$$a_{2m+3} = -2\frac{2m+2}{2m+3}\, a_{2m+1} = -4\frac{m+1}{2m+3}\, a_{2m+1}$$

$$a_2 = -\frac{1}{1}\, a_0$$

$$a_4 = -\frac{3}{2}\, a_2 = \frac{1.3}{1.2}\, a_0$$

$$a_6 = -\frac{5}{3}\, a_4 = -\frac{1.3.5}{1.2.3}\, a_0$$

Example (Continue ...)

$$a_{2m} = (-1)^m \frac{1.3.5.\ldots.(2m-1)}{m!}\, a_0$$

$$= (-1)^m \frac{\prod_{j=1}^{m}(2j-1))}{m!}\, a_0$$

$$a_{2m+3} = -4\frac{m+1}{2m+3}\, a_{2m+1}$$

$$a_3 = -4\frac{1}{3}\, a_1$$

$$a_5 = -4\frac{2}{5}\, a_3 = 4^2\frac{1.2}{3.5}\, a_1$$

$$a_7 = -4\frac{3}{7}\, a_5 = -4^3\frac{1.2.3}{3.5.7}\, a_1$$

$$a_{2m+1} = (-1)^m 4^m \frac{m!}{\prod_{j=1}^{m}(2j+1)}\, a_1$$

Example (Continue ...)

We can write the solution

$$y = \sum_{0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary scalars and

$$y_1(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m} (2j-1)}{m!} x^{2m}$$

$$y_2(x) = \sum_{m=0}^{\infty} (-1) \frac{4^m m!}{\prod_{j=1}^{m} (2j+1)} x^{2m+1}$$

Since $P_0(x) = 1 + 2x^2$ has complex zeros $\frac{\pm\iota}{\sqrt{2}}$, the power series solution converges in the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. □

Example

Find the coefficients $a_0, \ldots, a_6$ in the series solution

$$y = \sum_{0}^{\infty} a_n x^n$$

of the IVP

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0$$

with

$$y(0) = -1,\ y'(0) = -2.$$

Zeros of $P_0(x) = 1 + x + 2x^2$ are $\frac{1}{4}(-1 \pm \iota\sqrt{7})$ whose absolute values are $1/\sqrt{2}$. Hence the series solution to the IVP converges on the interval $\left(\dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.

Example (Continue . . .)

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = \sum_0^\infty b_n x^n = 0$$

$$\begin{aligned} b_n = {} & (n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2n(n-1)a_n \\ & +(n+1)a_{n+1} + 7na_n + 2a_n = 0 \end{aligned}$$

that is

$$(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + (2n^2 + 5n + 2)a_n = 0$$

Since $2n^2 + 5n + 2 = (n+2)(2n+1)$,

$$a_{n+2} = -\frac{n+1}{n+2}\, a_{n+1} - \frac{2n+1}{n+1}\, a_n \qquad n \geq 0$$

Example (Continue ...)

$$a_{n+2} = -\frac{n+1}{n+2}\,a_{n+1} - \frac{2n+1}{n+1}\,a_n \qquad n \geq 0$$

From the initial conditions $y(0) = -1,\ y'(0) = -2$ we get

$$a_0 = y(0) = -1, \quad a_1 = y'(0) = -2$$

$$a_2 = -\frac{1}{2}a_1 - a_0 = 2$$

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = \frac{5}{3}$$

Check that

$$y(x) = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 + \dots$$