MA-207 Differential Equations II

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References

Elementary differential equations with boundary value problems by William F. Trench (available online)

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Differential Equations with Applications and Historical Notes
by George F. Simmons

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$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \qquad a_i \in \mathbb{R}$$

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$$\frac{x^3 + 3x + 2}{x^5 + 2x^3 + 5},$$

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$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \ldots + P_1(x)y + P_0(x) = 0$$

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3. exponential functions, for example $e^x, \ \log x$

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Thus

$$y = tan\left[\frac{xe^{1/x^2} + \tan^{-1}(1+x^2) + \sqrt{x^2+3}}{\sin x \cos 2x - \sqrt{\log x} + x^{3/2}}\right]^{1/3}$$

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Beyond elementary functions lie the special functions, for example, Gamma function, Beta function, Riemann zeta function etc.

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This is one of the millennium problems and has a prize of 1 million US dollars.

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$$y'' + y = 0$$

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Let $y_1(x)$ be one solution of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x), q(x)$ continuous. Then we can try to use the method of variation of parameters to find another linearly independent solution, that is, put

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Let us review power series, which is used throughout in this course.

Definition (Power series)

For real numbers $x_0, a_0, a_1, a_2, \ldots$, an infinite series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n := a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots.$$

is called a power series in $x - x_0$ with center x_0.

For a real number x_1, if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x_1 - x_0)^n$$

exists and is finite, then we say the power series converges at the point $x = x_1$. In this case, the sum of the series is the value of the limit.

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If the series does not converge at x_1, that is, either limit does not exist or it is $\pm\infty$, then we say the power series diverges at x_1.

Theorem

For any power series,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

exactly one of these statements is true.

1. *The power series converges only for* $x = x_0$.
2. *The power series converges for all values of* x.
3. *There is a positive number* $0 < R < \infty$ *such that the power series converges if* $|x - x_0| < R$ *and diverges if* $|x - x_0| > R$.

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Question. How to compute the radius of convergence?

Theorem

- *(Ratio test) If* $a_n \neq 0$ *for all* n *and*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

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For $L = 0$*, we get* $R = \infty$ *and for* $L = \infty$*, we get* $R = 0$.

Theorem

Let $R > 0$ be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$.

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The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.

Example

Find the radius of convergence and interval of convergence (if $R > 0$) of the following three series

$$(i)\ \sum_{0}^{\infty} n!x^n \quad (ii)\ \sum_{10}^{\infty}(-1)^n \frac{x^n}{n^n} \quad (iii)\ \sum_{0}^{\infty} 2^n n^3 (x-1)^n$$

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$$(i)\ \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} (n+1) = \infty$$

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Interval of convergence : in case (ii) $(-\infty, \infty)$ and in case (iii) $(1/2, 3/2)$

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- *f is infinitely differentiable $\forall\ x \in (x_0-R, x_0+R)$.*

- *The successive derivatives of f can be computed by differentiating the power series term-by-term, that is*

$$f'(x) = \sum_{n=0}^{\infty} n a_n(x-x_0)^{n-1} \qquad \dots$$

$$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\dots(n-k+1)\, a_n(x-x_0)^{n-k}$$

Theorem (continued ...)

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• *We can determine the coefficients* a_n *(in terms of derivatives of* f *at* x_0*) as*

$$f(x_0) = a_0, \quad f'(x_0) = a_1, \quad f''(x_0) = 2a_2, \ldots$$

In general,

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

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- *We can also integrate the function* $f(x) = \sum_0^\infty a_n(x - x_0)^n$ *term-wise that is if* $[a, b] \subset (x_0 - R, x_0 + R)$, *then*

$$\int_a^b f(x)\, dx = \sum_{n=0}^\infty a_n \int_a^b (x - x_0)^n\, dx = \sum_0^\infty \frac{a_n}{n+1}(x - x_0)^{n+1}$$

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$$= \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

Theorem

(i) Power series representation of f in an open interval I containing x_0 is unique, that is, if

$$f(x) = \sum_0^\infty a_n (x - x_0)^n = \sum_0^\infty b_n (x - x_0)^n$$

for all $x \in I$, then $a_n = b_n \ \forall \ n$.

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Proof. (i)

$$a_n = \frac{f^{(n)}(x_0)}{n!} = b_n \qquad \text{for all} \quad n.$$

It is clear that (ii) follows from (i).

Algebraic operations on power series

Definition

If $$f(x) = \sum_0^\infty a_n (x - x_0)^n \qquad g(x) = \sum_0^\infty b_n (x - x_0)^n$$

have radius of convergence R_1 and R_2 respectively, then

$$c_1 f(x) + c_2 g(x) := \sum_0^\infty (c_1 a_n + c_2 b_n)(x - x_0)^n$$

has radius of convergence $\boxed{R \geq \min\{R_1, R_2\}}$ for $c_1, c_2 \in \mathbb{R}$.

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Further, we can multiply the series as if they were polynomials, that is

$$f(x)g(x) = \sum_0^\infty c_n (x - x_0)^n; \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

It also has radius of convergence $R \geq \min \{R_1, R_2\}$.

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$$= \sum_{n=0}^{\infty}\frac{1}{2}\left[1+(-1)^n\right]\frac{x^n}{n!}$$

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$$\begin{aligned}
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&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{1}{2}\left[1 + (-1)^n\right]\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}
\end{aligned}$$

Example

Find the power series expansion for $\cosh x$ in terms of powers of x^n.

$$\begin{aligned}\cosh x &= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{1}{2}\left[1+(-1)^n\right]\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\end{aligned}$$

Since radius of convergence for Taylor series of e^x and e^{-x} are ∞, the power series expansion of $\cosh x$ is valid on $\mathbb{R}$.

Shifting the summation index

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In general, $\left[\sum_{n=n_0}^{\infty} b_n (x - x_0)^{n-k} = \sum_{n=n_0-k} b_{n+k} (x - x_0)^n \right]$

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&= \sum_{n=0}^{\infty} \left[(n+1)n a_{n+1} - (n+2)(n+1)a_{n+2}\right] x^n
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- Find necessary and sufficient conditions on the coefficients a_n's, so that $y(x)$ is a solution of the ODE

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Example (Continue ...)

Solution. Write the ODE in $(x-1)$, that is

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that is a_n's satisfy the following recursive relation

$$(n+1)n a_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n = 0$$

for all n.

Definition

If a function $f(x)$ is infinitely differentiable at x_0, then the Taylor series of f at x_0 is defined as the power series

$$TS\, f|_{x_0} := \sum_{0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

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The function $$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

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Hence the Taylor series of f at 0 is the constant function taking value 0.

Therefore Taylor series of f at 0 does not converge to function $f(x)$ on any open interval around 0.

Definition

Suppose

- $f(x)$ is infinitely differentiable at x_0; and
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The function $$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is not analytic at 0. Here 2nd condition fails.
However, f is analytic at all $x \neq 0$.

Theorem (Analytic functions)

1. *If* $f(x)$ *and* $g(x)$ *are analytic at* x_0*, then* $f(x) \pm g(x)$ $f(x)g(x)$ $f(x)/g(x)$ *(if* $g(x_0) \neq 0$*) are analytic at* x_0*.*

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If there were such a power series, then by uniqueness, it has to be the power series expansion of $h(x)$ around 0, which is

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In fact, for any x_0, there is no power series around x_0 which represents $h(x)$ everywhere.

Theorem

Let

$$F(x) = \frac{N(x)}{D(x)} \quad \left(\textit{example} \;\; F(x) = \frac{x^3 - 1}{x^2 + 1}\right)$$

be a rational function, where $N(x)$ and $D(x)$ are polynomials without any common factors, that is they do not have any common (complex) zeros. Let $\alpha_1, \ldots, \alpha_r$ be distinct complex zeros of $D(x)$.

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If x_0 is different from $\{\alpha_1, \ldots, \alpha_r\}$, then the radius of convergence R of the Taylor series of F at x_0

$$TS\,F_{x_0} = \sum_{0}^{\infty} \frac{F^{(n)}(x_0)}{n!}(x - x_0)^n$$

is given by

$$R = \mathit{min}\{|x_0 - \alpha_1|, |x_0 - \alpha_2|, \ldots, |x_0 - \alpha_r|\}$$

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If

$$F(x) = \frac{N(x)}{D(x)} = \frac{(2+3x)}{(4+x)(9+x^2)}$$

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Theorem (Existence Theorem)

If $p(x)$ and $q(x)$ are analytic functions at x_0, then every solution of

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In most applications, $p(x)$ and $q(x)$ are rational functions, that is quotient of polynomial functions.

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and the series will have ∞ radius of convergence.
Since

$$y'' = \sum_{2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

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Therefore,

$$a_2 = \frac{-1}{2.1} a_0, \quad a_4 = \frac{-1}{4.3} a_2 = \frac{1}{4!} a_0 \ \ldots \ a_{2n} = (-1)^n \frac{1}{(2n)!} a_0$$

$$a_3 = \frac{-1}{3.2} a_1, \quad a_5 = \frac{-1}{5.4} a_3 = \frac{1}{5!} a_1 \ \ldots \ a_{2n+1} = (-1)^n \frac{1}{(2n+1)!} a_1$$

Example (Continue ...)

Define

$$y_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \qquad (a_0 = 1, a_1 = 0)$$

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We don't need to check the series for converges, since the existence theorem guarantees that the series converges for all x.

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6. From the recursion formula, obtain (linearly independent) solutions $y_1(x)$ and $y_2(x)$. The general solution then looks like $y(x) = a_1 y_1(x) + a_2 y_2(x)$.

The following ODE's are classical:

- Bessel's equation :

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It occurs in problems displaying spherical symmetry, particularly in electromagnetism.

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When x_0 is a zero of $P_0(x)$, then x_0 is called a singular point of ODE. This case will be considered later.

Example

Find the power series in x for the general solution of

$$(1 + 2x^2)y'' + 6xy' + 2y = 0$$

Solution. Note that 0 is not a zero of $P_0(x) = 1 + 2x^2$, hence the series solution in powers of x exists.

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Example (Continue . . .)

$$\implies a_{n+2} = -\frac{2n^2+4n+2}{(n+2)(n+1)}\,a_n = -2\frac{n+1}{(n+2)}\,a_n \quad n \geq 0$$

Since indices on left and right differ by 2, we write separately for $n = 2m$ and $n = 2m+1, \;\; m \geq 0$, so

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$$a_6 = -\frac{5}{3}\,a_4 = -\frac{1.3.5}{1.2.3}\,a_0$$

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$$a_7 = -4\frac{3}{7}\, a_5 = -4^3\frac{1.2.3}{3.5.7}\, a_1$$

Example (Continue ...)

$$a_{2m} = (-1)^m \frac{1.3.5.\ldots.(2m-1)}{m!} a_0$$

$$= (-1)^m \frac{\prod_{j=1}^{m}(2j-1))}{m!} a_0$$

$$a_{2m+3} = -4\frac{m+1}{2m+3} a_{2m+1}$$

$$a_3 = -4\frac{1}{3} a_1$$

$$a_5 = -4\frac{2}{5} a_3 = 4^2 \frac{1.2}{3.5} a_1$$

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$$a_{2m+1} = (-1)^m 4^m \frac{m!}{\prod_{j=1}^{m}(2j+1)} a_1$$

Example (Continue ...)

We can write the solution

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Since $P_0(x) = 1 + 2x^2$ has complex zeros $\frac{\pm\iota}{\sqrt{2}}$, the power series solution converges in the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. □

Example

Find the coefficients $a_0, \ldots, a_6$ in the series solution

$$y = \sum_{0}^{\infty} a_n x^n$$

of the IVP

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0$$

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Zeros of $P_0(x) = 1 + x + 2x^2$ are $\frac{1}{4}(-1 \pm \iota\sqrt{7})$ whose absolute values are $1/\sqrt{2}$. Hence the series solution to the IVP converges on the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Example (Continue ...)

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = \sum_{0}^{\infty} b_n x^n = 0$$

$$b_n = (n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2n(n-1)a_n$$

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Since $2n^2 + 5n + 2 = (n+2)(2n+1)$,

$$a_{n+2} = -\frac{n+1}{n+2}\,a_{n+1} - \frac{2n+1}{n+1}\,a_n \qquad n \geq 0$$

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From the initial conditions $y(0) = -1,\; y'(0) = -2$ we get

$$a_0 = y(0) = -1, \quad a_1 = y'(0) = -2$$

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Check that

$$y(x) = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 + \dots$$