MA-207 Differential Equations II

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2. Power series defines a nice function in its interval of convergence
3. Suppose we are given an ODE: $y'' + p(x)y' + q(x)y = 0$, and $p(x)$ and $q(x)$ are analytic (given by power series) in an interval I around x_0, then the solution y is also analytic on I.

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4. We can compute the two independent solutions, to an ODE as above, by plugging in a power series into the ODE and getting recursive relation for coefficients.

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Equating the coefficient of x^n in the resulting equation, we get the recursive relation

$$(n+2)(n+1)a_{n+2} - n(n+1)a_n + p(p+1)a_n = 0, \quad n \geq 0$$

$$\implies \qquad a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)}\, a_n$$

$$\Longrightarrow \qquad a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)}\, a_n$$

Let us set $\boxed{a_0 = 1 \text{ and } a_1 = 0}$ in the recursion formula to find a first solution.

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The solution is given by (note it is an even function)

$$y_1(x) := a_0 \left[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p+1)(p-2)(p+3)}{4!} x^4 + \dots \right]$$

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Thus, if p is an integer then exactly one solution is a polynomial and the other is an infinite power series.

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Let us write down few Legendre polynomials.

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= (1-3x^2)(\tfrac{-1}{2}) &&= \tfrac{1}{2}(3x^2-1) \\
P_3(x) &= (x-\tfrac{5}{3}x^3)(\tfrac{-3}{2}) &&= \tfrac{1}{2}(5x^3-3x) \\
P_4(x) &= (1-10x^2+\tfrac{35}{3}x^4)(\tfrac{3}{8}) &&= \tfrac{1}{8}(35x^4-30x^2+3) \\
P_5(x) &= (x-\tfrac{14}{3}x^3+\tfrac{21}{5}x^5)(\tfrac{15}{8}) &&= \tfrac{1}{8}(63x^5-70x^3+15x)
\end{aligned}$$

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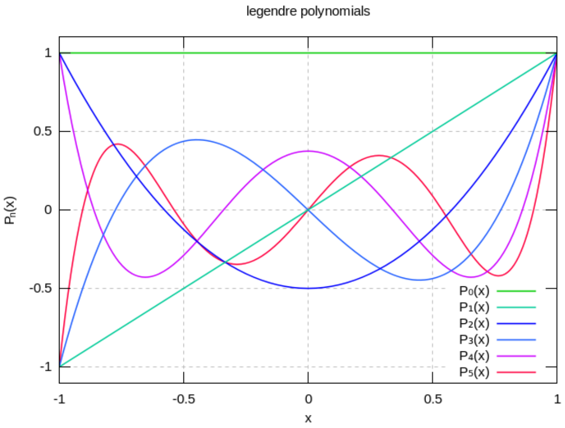

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To answer this question we need some linear algebra.

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- addition

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- scalar multiplication

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A vector space with an inner product is called an inner product space.

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Thus, $$a_j = \frac{\langle v, e_j \rangle}{\langle e_j, e_j \rangle}$$ □

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- $\|0\| = 0$ and $\|v\| > 0$ if $v \neq 0$
- $\|v + w\| \leq \|v\| + \|w\|$
- $\|av\| = |a|\|v\|$

for all $v, w \in V$ and $a \in \mathbb{R}$.

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&= \langle v, v \rangle + \langle w, w \rangle \\
&= \|v\|^2 + \|w\|^2
\end{aligned}$$

□

Pythagoras theorem

Theorem

For **orthogonal** *vectors v and w in any inner product space V,*

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Proof.

$$\begin{aligned}\|v + w\|^2 &= \langle v + w, v + w\rangle \\ &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= \langle v, v\rangle + \langle w, w\rangle \\ &= \|v\|^2 + \|w\|^2\end{aligned}$$

□

More generally, for any orthogonal system $\{v_1, \ldots, v_n\}$

$$\|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2$$

The vector space of polynomials

The set of all polynomials in the variable x is a vector space denoted by $\mathcal{P}(x)$.

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The norm of a polynomial is by definition $\langle f, f \rangle$

$$\|f\| := \left(\int_{-1}^{1} f(x)f(x)dx \right)^{1/2}$$

Derivative transfer

Note that

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This will be referred to as derivative-transfer

Orthogonality of Legendre polynomials

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Since $P_m(x)$ is a polynomial of degree m, it follows that

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Theorem

We have

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

i.e. Legendre polynomials form an orthogonal basis for the vector space $\mathcal{P}(x)$ and

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

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The Legendre equation may be written as

$$((1 - x^2)y')' + p(p + 1)y = 0$$

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If $m \neq n$ we get

$$\int_{-1}^{1} P_m P_n = 0$$

Thus, P_m and P_n are orthogonal. □

Rodrigues formula

It only remains to show that $\|P_n(x)\|^2 = \dfrac{2}{2n+1}$.

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Consider the polynomial of degree n given by

$$y(x) = D^n(x^2-1)^n$$

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For $k < n$ consider the integral

$$\int_{-1}^{1} P_k(x)y(x) = \int_{-1}^{1} P_k(x)D(D^{n-1}(x^2-1)^n)$$

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Since $P_k(x)$ is a polynomial of degree k we get that $D^nP_k(x) = 0$.

Rodrigues formula

This forces that $y(x) = cP_n(x)$ for some nonzero constant c as we know that $P_k(x)$'s are orthogonal to each other.

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Thus, we can normalize our Legendre polynomials so that $P_m(1) = 1$. That is, take

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This is called Rodrigues formula.

Computing $\|P_n(x)\|$

Proof.

$$\int_{-1}^{1} P_n(x)P_n(x)\,dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

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$$I_n = \int_{-1}^{1}(1-x^2)^n\,dx = \int_{-1}^{1}(1-x^2)^n\frac{dx}{dx}dx$$

$$\overset{dt}{=} 2n\int_{-1}^{1}(1-x^2)^{n-1}x^2dx = -2nI_n + 2nI_{n-1}$$

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This completes the proof of the theorem.

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Fourier-Legendre series

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We can expand any square-integrable function $f(x)$ on $[-1,1]$ in a series of Legendre polynomials

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

This is called the Fourier-Legendre series (or simply the Legendre series) of $f(x)$.

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There are two issues here:

- Does the Fourier-Legendre series converge at x?
- If yes, then does it converge to $f(x)$?

A useful result in this direction is the Legendre expansion theorem:

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In particular, the series converges to $f(x)$ at every point of continuity x.

Example

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

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Recall, $P_1(x) = x$, so

$$c_1 = \frac{3}{2} \int_{-1}^{1} f(x) x\, dx = \frac{3}{2}$$

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By the Legendre expansion theorem, this series converges to $f(x)$ for $x \neq 0$ and to 0 for $x = 0$.

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Proof.

Write degree $\leq n$ polynomial $p(x) = \sum_{k=0}^{n} b_k P_k(x)$, then

proof continued . . .

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Clearly, I is minimum when $b_k = c_k$ for $k = 0, \ldots, n$.

Caution. If f has a power series expansion on $[-1, 1]$, then best "least square polynomial approximation" to $f(x)$ is not the partial sums of the power series, in general.

Some remarks

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1. Given an ODE of the type

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \qquad (*)$$

first convert it to the standard form

$$y'' + \frac{F_1(x)}{F_0(x)}y' + \frac{F_2(x)}{F_0(x)}y = 0 \qquad (**)$$

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Let

$$p(x) := \frac{F_1(x)}{F_0(x)} \qquad q(x) := \frac{F_2(x)}{F_0(x)}$$

Some remarks

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4. To find the solutions in a neighborhood of x_0, set $y(x) = \sum_{n \geq 0} a_n (x - x_0)^n$ into the ODE $(*)$ or $(**)$ and get recursive relations involving the a_n. Note that when you do this, the coefficient functions $(p(x), q(x), F_0(x), ..)$ have to be written as power series in $x - x_0$. Note that the recursive relation you get, will be same, irrespective of whether you choose equation $(*)$ or $(**)$.

Some remarks

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For example, the Legendre equation, in the open interval $(-1, 1)$ around $x_0 = 0$, the equation $(*)$ looks like

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$

while $(**)$ looks like

$$y'' - 2\Big(\sum_{n\geq 0} x^{2n+1}\Big)y' + p(p+1)\Big(\sum_{n\geq 0} x^{2n}\Big)y = 0$$

In this case it is clear that, we should choose $(*)$, as it will be easier to work with. This is what we did in class.