MA-207 Differential Equations II

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For example, $$\Gamma(-\frac{5}{2}) = \frac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}} = \frac{\Gamma(-\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})} = \frac{\Gamma(\frac{1}{2})(=\sqrt{\pi})}{(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})}$$

Further

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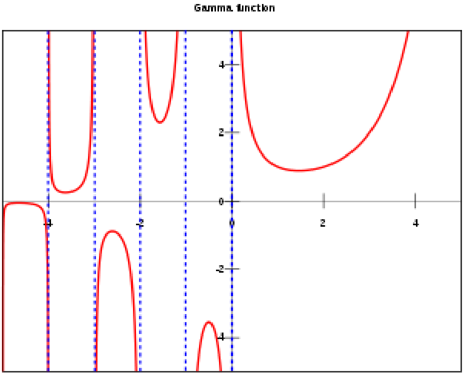
The graph of Gamma function is shown below.



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It is useful to rewrite

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$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty t^{-1/2} e^{-t} dt \\ &= 2\int_0^\infty e^{-s^2} ds \qquad \text{(use the substitution } t = s^2\text{)} \\ &= \sqrt{\pi}\end{aligned}$$

By translating,

$$
\begin{aligned}
\Gamma(1/2) & & &= \sqrt{\pi} & &\approx 1.772 \\
\Gamma(-1/2) &= \frac{\Gamma(1/2)}{-1/2} & &= -2\sqrt{\pi} & &\approx -3.545 \\
\Gamma(-3/2) &= \frac{\Gamma(-1/2)}{-3/2} & &= \frac{4}{3}\sqrt{\pi} & &\approx 2.363 \\
\Gamma(3/2) &= \frac{1}{2}\Gamma(1/2) & &= \frac{1}{2}\sqrt{\pi} & &\approx 0.886 \\
\Gamma(5/2) &= \frac{3}{2}\Gamma(3/2) & &= \frac{3}{4}\sqrt{\pi} & &\approx 1.329 \\
\Gamma(7/2) &= \frac{5}{2}\Gamma(5/2) & &= \frac{15}{8}\sqrt{\pi} & &\approx 3.323
\end{aligned}
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The second linearly independent solution of $(*)$ is called Bessel function of second kind.

For Frobenius solution, put $y = x^r \sum_{n=0}^{\infty} a_n(r)x^n \quad a_0 = 1$.

The indicial equation, i.e. coefficient of x^r, for Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ is

$$I(r) = r(r-1) + r - p^2 = r^2 - p^2 = 0$$

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The roots are $r_1 = p$ and $r_2 = -p$.
For recurrence relations, equating coefficient of x^{n+r} to 0 (for $n \geq 1$) we get

$$[(r+n)^2 - p^2]a_n(r) + a_{n-2}(r) = 0 \qquad n \geq 2$$

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$$a_{2n}(r) = \frac{-1}{(r+2n)^2 - p^2}\, a_{2n-2}$$
$$= \frac{(-1)^n}{((r+2)^2 - p^2)((r+4)^2 - p^2)\ldots((r+2n)^2 - p^2)}$$

For Frobenius solution, set $r = p$ the larger root.

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$$= \frac{(-1)^n}{(2(2p+2))(4(2p+4))\dots(2n(2p+2n))}$$

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The solution $$y_1(x) = x^p \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)} x^{2n}$$

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Multiply $y_1(x)$ by $\dfrac{1}{2^p \Gamma(1+p)}$

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n} \quad x > 0.$$

This is called the Bessel function of first kind of order p.

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The Bessel function of order 0 is

$$\begin{aligned} J_0(x) &= \sum_{n\geq 0} \frac{(-1)^n}{n!n!} \left(\frac{x}{2}\right)^{2n} \\ &= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!3!}\left(\frac{x}{2}\right)^6 + \dots \end{aligned}$$

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The Bessel function of order 1 is

$$\begin{aligned} J_1(x) &= \sum_{n\geq 0} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1} \\ &= \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 + \dots \end{aligned}$$

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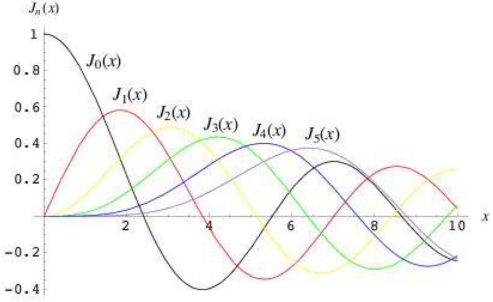

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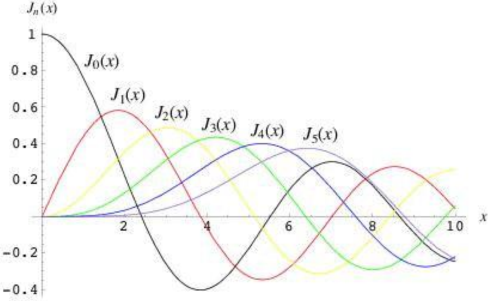

Further, they satisfy derivative identities similar to $\cos x$ and $\sin x$.

$$J_0'(x) = -J_1(x) \qquad\qquad [xJ_1(x)]' = xJ_0(x)$$

Second independent solution of Bessel equation

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- $2p$ is not an integer
- $2p$ is an odd positive integer
- $2p$ is an even positive integer
- $p = 0$

Case 1: $2p$ is not an integer.

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Solving the recursion

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Thus, $A_{2n}(-p) = 0$ and $A'_{2n}(-p) = a_{2n}(-p)$.
Thus, in this case we obtain that the second solution is

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If $n \geq p$, then

$$\begin{aligned} A_{2n}(-p) &= \frac{2(-1)^n}{2^{2n} n!(1-p)\dots(-1)\cdot 1\cdot 2\cdots(n-p)} \\ &= \frac{-2(-1)^{n-p}}{2^{2n} n!(p-1)!(n-p)!} \end{aligned}$$

Define

$$f(r) := \left(\prod_{i=1}^{p-1}((r+2i)^2 - p^2)\right)(r+3p)\left(\prod_{i=p+1}^{n}((r+2i)^2 - p^2)\right) \quad (*)$$

Then

$$A_{2n}(r)f(r) = (-1)^n$$

Differentiating the above and setting $r = -p$ we get

$$A'_{2n}(-p)f(-p) + A_{2n}(-p)f'(-p) = 0$$

Taking log and differentiating $(*)$ we get

$$f'(-p) = f(-p)\Big(\frac{1}{2p} + \sum_{i\in\{1,2,\ldots,n\}\setminus p} \frac{1}{2i} + \frac{1}{2(i-p)}\Big)$$

$$= f(-p)\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big),$$

where

$$H_0 = 0, \qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

Thus,

$$A'_{2n}(-p) = -A_{2n}(-p)\left(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\right)$$
$$= \frac{2(-1)^{n-p}}{2^{2n} n!(p-1)!(n-p)!}\left(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\right)$$

Thus, we get

$$y_2(x) = \sum_{n=0}^{p-1} \frac{1}{2^{2n} n!(p-n)!} x^{2n-p} +$$
$$\sum_{n \geq p} \frac{(-1)^{n-p}}{2^{2n} n!(p-1)!(n-p)!}\Big(H_n - H_{p-1} + H_{n-p}\Big) x^{2n-p} +$$
$$-\sum_{n \geq p} \frac{2(-1)^{n-p}}{2^{2n} n!(p-1)!(n-p)!} x^{2n-p} \log x$$

is a second solution.

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where $y_1(x) = J_0(x) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$ is Frobenius solution.

Summary of $p = 0$ and $p = 1/2$

For $p = 0$, two independent solutions are $J_0(x)$, which is a real analytic function for all $\mathbb{R}$, and

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Both exhibit singular behavior at 0. Near 0, $J_{1/2}(x)$ is bounded but does not have a bounded derivative, while $J_{-1/2}(x)$ is unbounded near 0.

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1. The above is a well defined power series once we know that the Gamma function never vanishes.
2. If $p \notin \{0, 1, 2, \ldots\}$ $J_p(x)$ and $J_{-p}(x)$ are the two independent solutions of the Bessel equation.
3. If $p \in \{0, 1, 2, \ldots\}$ then $J_{-p}(x) = (-1)^p J_p(x)$. Thus, in this case the second solution is not $J_{-p}(x)$.

Bessel identities

1. $$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$
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Add and subtract (3) and (4) to get (5) and (6).

Consequences of Bessel identities

Problem: Let $p > 0$. Show that between any two consecutive zeros of $J_p(x)$, there exists precisely one zero of $J_{p-1}(x)$ and precisely one zero of $J_{p+1}(x)$

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Repeating the above argument with the identity $[x^{-p} J_p(x)]' = -x^{-p} J_{p+1}(x)$, we get that $J_{p+1}(x)$ has a root in (c, d).

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Hence $a = 2$ and $c = 0$.

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For example,

- $J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$

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- $J_{\frac{5}{2}}(x) = \dfrac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$

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These functions are called spherical Bessel functions as they arise in solving wave equations in spherical coordinates.

Theorem (Liouville)

$J_{m+\frac{1}{2}}(x)$'s are the only elementary Bessel functions.

Remark. Integrating some of the Bessel identities we get

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$
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It is rarely possible to solve 2nd order linear ODE

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Then how do we understand the nature and properties of solutions.

It is surprising that we can obtain quite a bit of information about the solution from the ODE itself.

Theorem (Sturm separation theorem)

If $y_1(x)$ and $y_2(x)$ are linearly independent solns of

$$y'' + P(x)y' + Q(x)y = 0$$

P, Q continuous on (a, b). Then
(1) $y_1(x)$ and $y_2(x)$ have no common zero in (a, b).
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If $y_1(x)$ and $y_2(x)$ are linearly independent solns of

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In particular, since y_1 and y_2 are linearly independent, the Wronskian is nonzero and so it never vanishes. This proves (1).

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If $y_2(x)$ had two zeros in the interval $x_1 < \alpha < \beta < x_2$, then by the same reasoning, y_1 will have a zero in (α, β), which contradicts the assumption that x_1 and x_2 are successive zeros of y_1.

As a consequence, if y_1 and y_2 are linearly independent solution of $y'' + P(x)y' + Q(x)y = 0$, P, Q continuous on (a, b) then the number of zeros of y_1 and y_2 on (a, b) differ by at most 1.

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Also note that we need $P(x)$ to be once differentiable.

Theorem

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ on the interval (α, β), with $q(x)$ continuous. Let $[a, b] \subset (\alpha, \beta)$ be a finite interval. Then $u(x)$ has at most finite number of zeros in $[a, b]$. Hence if $u(x)$ has infinitely many zeros on $(0, \infty)$, then the set of zeros of $u(x)$ are not bounded.

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$u(x_0) = \lim_{x_n \to x_0} u(x_n) = 0$ (u is continuous) and

$$u'(x_0) = \lim_{x_n \to x_0} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0$$

This contradicts the fact that the Wronskian at x_0 is nonzero. $\square$

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But this is not possible as u' is increasing on (x_0, x_1). $\square$

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Let $u(x)$ *be a non-trivial solution of* $u'' + q(x)u = 0$ *Let* $q(x)$ *be continuous and* $q(x) > 0$ *for all* $x > x_0 > 0$.

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If we show that $u'(x_2) < 0$ for some $x_2 > x_1$, then we get for $x > x_2$

$$\begin{aligned} u(x) = \int_{x_2}^{x} u'(t)dt + u(x_2) &\leq \int_{x_2}^{x} u'(x_2)dt + u(x_2) \\ &\leq u'(x_2)(x - x_2) + u(x_2) \end{aligned}$$

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Thus, $u'(x) = -u(x)v(x)$ and this shows that $u'(x) < 0$ for x large.

Theorem

In Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ *Substituting* $u(x) = \sqrt{x}y(x)$, *we get*

$$u'' + \left[1 + \frac{1 - 4p^2}{4x^2}\right] u = 0$$

$q(x) = 1 + \dfrac{1 - 4p^2}{4x^2}$ *is continuous and* $q(x) > 0$ *for* $x > x_0 > 0$. *Further,*

$$\int_{x_0}^{\infty} \left(1 + \frac{1 - 4p^2}{4x^2}\right) dx = \infty$$

By previous theorem, $u(x)$, *hence any Bessel function has infinitely many zeros on* $(0, \infty)$.

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Question. What is the limit of $x_{n+1} - x_n$ as $n \to \infty$?

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We will need the Sturm comparison theorem.

Theorem (Sturm Comparison theorem)

Let $y(x)$ be a non-trivial solutions of

$$y'' + q(x)y = 0$$

and $z(x)$ be a non-trivial solutions of

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Zeros of $y(x)$ are $\pi/2$ apart and that of $z(x)$ are π apart.

Proof of Sturm Comparison theorem.

Let $x_1 < x_2$ be consecutive zeros of $z(x)$.

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We may assume $z(x) > 0$ and $y(x) > 0$ on (x_1, x_2). Hence $z'(x_1) > 0$ and $z'(x_2) < 0$.

Proof of Sturm Comparison theorem.

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Integrating from x_1 to x_2, we get

$$W(x_2) - W(x_1) > 0 \implies W(x_2) > W(x_1)$$

But $W(x_1) = y(x_1)z'(x_1) > 0$ and $W(x_2) = y(x_2)z'(x_2) < 0$, a contradiction. $\square$

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Let $y_p(x)$ *be a non-trivial solution of Bessel equation. Then we get ...*

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- $\boxed{p < 1/2} \implies$ *Between any two roots of* $\alpha \cos x + \beta \sin x$ *there is a root of* $y_p(x)$.
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It is a trivial check that f satisfies the differential equation

$$f'' + r(x)f = 0 \qquad\qquad r(x) := q(x - b + a)$$

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As $x_0 < c$ we get that $2b - a < c$, that is, $b - a < c - b$.

Next we claim that the difference between any two successive roots of u is strictly less than π.

If not, then let $a < b$ be successive roots such that $b - a \geq \pi$

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Thus, we have proved that if $\{x_n\}$ are the roots of u in increasing order, then the difference $x_{n+1} - x_n$ is strictly increasing and bounded above by π.

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Thus, we have proved that if $\{x_n\}$ are the roots of u in increasing order, then the difference $x_{n+1} - x_n$ is strictly increasing and bounded above by π.

Next let us show that these differences converge to π. If not, then $(x_{n+1} - x_n) \to \gamma < \pi$. Choose $1 < \delta$, sufficiently close to 1 such that $\gamma < \frac{\pi}{\delta} < \pi$.

The function $q(x)$ is decreasing to 1. Therefore, there is a $x_0 \in \mathbb{R}$, sufficiently large, such that $q(x_0) < \delta^2$. Apply Sturm's comparison on the interval (x_0, ∞) to the differential equations
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Thus, between any two roots of u there is a root of z. Let a and b be two consecutive roots of u such that $x_0 < a < b$. Since $b - a < \gamma < \frac{\pi}{\delta}$, find a' and b' such that $x_0 < a' < a < b < b'$ and $b' - a' = \frac{\pi}{\delta}$.

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Find α and β such that the function $\alpha \cos \delta\, x + \beta \sin \delta\, x$ vanishes at a'. This function is a solution to the ODE $z'' + \delta^2 z = 0$. The next root of this function is at $a' + \frac{\pi}{\delta} = b'$. Thus, we get a contradiction to Sturm's theorem which says that there is a root of this function in the interval (a, b).

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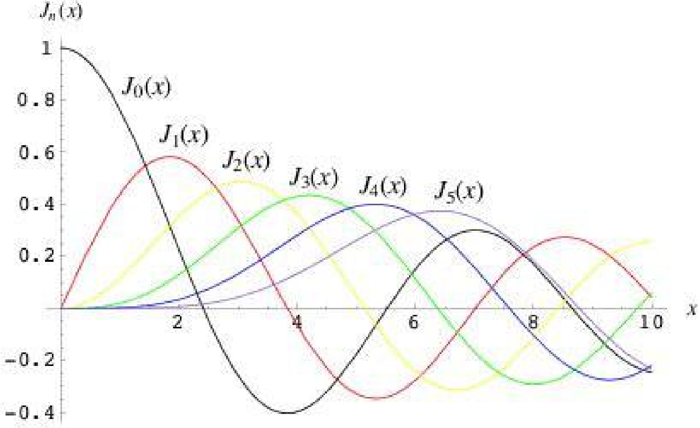
Similarly, we can prove that if $p > 1/2$ *then the sequence of difference of roots of* u *is decreasing and tends to* π.


$J_n(x)$
1
0.8
0.6
0.4
0.2
-0.2
-0.4
2
4
6
8
10
x
$J_0(x)$
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The first few zeroes of Bessel functions are tabulated below.

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
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Question. Why are we concerned with zeros of Bessel function $J_p(x)$?

It is often required in mathematical physics to expand a given function in terms of Bessel functions.

Simplest and most useful expansions are of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_{p,n} x) = a_1 J_p(\lambda_{p,1} x) + a_2 J_p(\lambda_{p,2} x) + \dots$$

where $f(x)$ is defined on, (say) $[0,1]$, and $\lambda_{p,n}$'s are zeros of Bessel function $J_p(x)$, $p \geq 0$.

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Remark: For a scalar a, the scaled Bessel functions $J_p(ax)$ are solutions of

$$x^2 y'' + xy' + (a^2 x^2 - p^2) y = 0$$

known as scaled Bessel equation.

Orthogonality

Define an inner product on functions on $[0,1]$ by

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This is similar to the previous inner product except that $f(x)g(x)$ is now multiplied by x and the interval of integration is from 0 to 1.

We call a function on $[0,1]$ square integrable with respect to this inner product if

$$\int_0^1 x f(x)^2 dx < \infty$$

The multiplying factor x is called a weight function.

Fix $p \geq 0$. Let $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ denote the set of zeros of $J_p(x)$ on $(0, \infty)$.

Theorem

The set of scaled Bessel functions

$$\{J_p(\lambda_{p,1}x), J_p(\lambda_{p,2}x), \ldots\}$$

form an orthogonal family w.r.t. above inner product, i.e. $\langle J_p(\lambda_{p,k}x), J_p(\lambda_{p,l}x)\rangle :=$

$$\int_0^1 xJ_p(\lambda_{p,k}x)J_p(\lambda_{p,l}x)\,dx = \begin{cases} \frac{1}{2}[J_{p+1}(\lambda_{p,k})]^2 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

Theorem

Fix $p \geq 0$ *and* $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ *: zeros of* $J_p(x)$ *on* $(0, \infty)$*. Any square-integrable function* $f(x)$ *on* $[0, 1]$ *can be expanded in a series of scaled Bessel functions* $J_p(\lambda_{p,n} x)$ *as*

$$f(x) = \sum_{n \geq 1} c_n J_p(\lambda_{p,n} x)$$

where

$$c_n = \frac{2}{[J_{p+1}(\lambda_{p,n})]^2} \int_0^1 x\, f(x)\, J_p(\lambda_{p,n} x)\, dx$$

This is Fourier-Bessel series of $f(x)$ *for parameter* p*.*

Example. Let us compute the Fourier-Bessel series (for $p = 0$) of $f(x) = 1$ in the interval $0 \leq x \leq 1$.

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Use $\frac{d}{dx}\big(x^p J_p(x)\big) = x^p J_{p-1}(x)$ for $p = 1$.

$$\int_0^1 x\, J_0(\lambda_{0,n} x)\, dx = \frac{1}{\lambda_{0,n}} x\, J_1(\lambda_{0,n} x)\Big|_0^1 = \frac{J_1(\lambda_{0,n})}{\lambda_{0,n}}$$

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By next theorem, this converges to 1 for $0 < x < 1$.

Convergence in norm

Fourier-Bessel series converges to $f(x)$ in norm, i.e.

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For pointwise convergence, we have

Bessel expansion theorem

Assume f and f' have at most a finite number of jump discontinuities in $[0, 1]$, then the Bessel series converges for $0 < x < 1$ to

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at $x = 0$, if $p = 0$ then it converges to $f(0_+)$.
at $x = 0$, if $p > 0$ then it converges to 0.

Proof of orthogonality of scaled Bessel functions

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If a, b are positive scalars, then $u(x) = J_p(ax)$ and $v(x) = J_p(bx)$ satisfies

$$u'' + \frac{1}{x}u' + \left(a^2 - \frac{p^2}{x^2}\right)u = 0$$

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Multiply by v and u resp. and subtract, we get

$$(vu'' - uv'') + \frac{1}{x}(vu' - uv') + (a^2 - b^2)uv = 0$$

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Multiply by v and u resp. and subtract, we get

$$(vu'' - uv'') + \frac{1}{x}(vu' - uv') + (a^2 - b^2)uv = 0$$

$$(u'v - v'u)' + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv$$

$$(x(u'v - v'u))' = (b^2 - a^2)xuv$$

$$(b^2 - a^2) \int_0^1 xuv \, dx = [x(u'v - v'u)]\Big|_0^1 = (u'v - v'u)(1)$$

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So if $a = \lambda_{p,k}$ and $b = \lambda_{p,l}$ are **distinct**, then

$$\int_0^1 xJ_p(\lambda_{p,k}x)J_p(\lambda_{p,l}x)\, dx = 0$$

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To compute the norm of $J_p(\lambda_{p,k}x)$, consider

$$\begin{aligned} 2x^2u'\left[u'' + \frac{1}{x}u' + (a^2 - \frac{p^2}{x^2})u\right] &= 0 \\ = [x^2u'^2 + (a^2x^2 - p^2)u^2]' - 2a^2xu^2 & \end{aligned}$$

Integrate on $[0,1]$,

$$2a^2 \int_0^1 xu^2\, dx = \left[x^2 u'^2 + (a^2x^2 - p^2)u^2\right]\Big|_0^1$$

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Put $a = \lambda_{p,k}$ to get

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Thus,

$$\int_0^1 xJ_p(\lambda_{p,k}x)^2\,dx = \frac{1}{2}J_p'(\lambda_{p,k})^2 = \frac{1}{2}J_{p+1}(\lambda_{p,k})^2$$

for last equality, use $x = \lambda_{p,k}$ in $J_p'(x) - \dfrac{p}{x}J_p(x) = J_{p+1}(x)$