MA-207 Differential Equations II

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Eigen Value problems $y'' + \lambda y = 0$

We will develop Fourier series representations of functions that will be used to solve PDE considered later.

Consider the following Boundary Value Problems (BVP), where $\lambda \in \mathbb{R}$ and $L > 0$.

1. Problem 1. $y'' + \lambda y = 0 \qquad y(0) = 0, \qquad y(L) = 0.$
2. Problem 2. $y'' + \lambda y = 0 \qquad y'(0) = 0, \qquad y'(L) = 0.$
3. Problem 3. $y'' + \lambda y = 0 \qquad y(0) = 0, \qquad y'(L) = 0.$
4. Problem 4. $y'' + \lambda y = 0 \qquad y'(0) = 0, \qquad y(L) = 0.$
5. Problem 5. $y'' + \lambda y = 0 \quad y(-L) = y(L),\ y'(-L) = y'(L).$

The boundary condition in problem 5 is called periodic.

Eigenvalue problem $y'' + \lambda y = 0$

Question. For what values of λ does the problem have a non-trivial solutions and what are the solutions?

Any λ for which the problem (1-5) has a non-trivial solution is called an eigenvalue of that problem

Non-trivial solutions for an eigenvalue λ are called λ-eigenfunction, or eigenfunction associated with λ.

A non-zero constant multiple of a λ-eigenfunction is again a λ-eigenfunction.

Problems $1 - 5$ are called eigenvalue problems. Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions.

Theorem

1. *Problems* $1-5$ *have no negative eigenvalues.*
2. $\lambda = 0$ *is an eigenvalue of Problems* 2 *and* 5 *with associated eigenfunctions* $y_0 = 1$.
3. $\lambda = 0$ *is not an eigenvalue of Problems* $1, 3$ *and* 4.

Proof.

Let us prove first two; third is left as an exercise.

Suppose $\lambda < 0$. Let us write $\lambda = -a^2$.

Rewrite the differential equation as $y'' = a^2 y$. The general solution to this is $y(x) = Ce^{ax} + De^{-ax}$. In problem 1 we have the condition $y(0) = y(L) = 0$. This forces that $C + D = 0$ and $Ce^{aL} + De^{-aL} = 0$. One checks easily that this forces $C = D = 0$.

In problem 2 we have the condition that $y'(0) = y'(L) = 0$. This gives $aC - aD = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Since $a \neq 0$, this forces $C = D = 0$.

Proof.

In problem 3 we have the conditions $y(0) = y'(L) = 0$. This gives $C + D = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Again this forces $C = D = 0$.

Similarly, do the other problems.

Now consider the second statement in the theorem. If $\lambda = 0$, the clearly, the solution has to be of the form $y(x) = ax + b$.

In problem 2 we have $y'(0) = y'(L) = 0$, and so $a = 0$. Thus, $y(x) = \text{constant}$ is the solution in this case.

In problem 5, we have $y(-L) = y(L)$, that is, $-aL + b = aL + b$. This forces that $a = 0$. Thus, in this case too $y(x) = \text{constant}$. □

Eigenvalue Problem 1

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$y_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots .$$

There are no other eigenvalues.

Proof.

Any eigen value must be positive (by previous theorem).

If y is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y(0) = 0 \implies c_1 = 0$$

$$\implies y(x) = c_2 \sin \sqrt{\lambda} x \quad \text{with} \quad c_2 \neq 0$$

$$y(L) = 0 \implies \sin \sqrt{\lambda} L = 0 \implies \sqrt{\lambda} L = n\pi$$

$$\implies \lambda_n = \frac{n^2 \pi^2}{L^2}$$

is an eigenvalue with an associated eigenfunction

$$y_n = \sin \frac{n\pi x}{L}$$

□

Eigenvalue Problem 2

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y'(0) = 0, \quad y'(L) = 0$$

has an eigenvalue $\lambda_0 = 0$ *with eigenfunction* $y_0 = 1$

and infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$y_n = \cos\frac{n\pi x}{L} \quad n = 1, 2, \ldots.$$

There are no other eigenvalues.

Proof. Similar to the proof of Problem 1, hence is left as an exercise.

Eigenvalue Problem 3

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \quad y'(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{(2n+1)^2 \pi^2}{4L^2}$$

with associated eigenfunctions

$$y_n = \sin \frac{(2n+1)\pi x}{2L}, \quad n = 0, 1, 2, \ldots .$$

There are no other eigenvalues.

Proof.

Any eigen value must be positive (by previous theorem).

If y is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

$$y(0) = 0 \implies c_1 = 0$$

$$\implies y(x) = c_2 \sin\sqrt{\lambda}x \quad \text{with} \quad c_2 \neq 0$$

$$y'(L) = 0 \implies \sqrt{\lambda}\cos\sqrt{\lambda}L = 0 \implies \sqrt{\lambda}L = \tfrac{2n+1}{2}\pi$$

$$\implies \lambda_n = \frac{(2n+1)^2\pi^2}{4L^2}$$

is an eigenvalue with an associated eigenfunction

$$y_n = \sin\frac{(2n+1)\pi x}{2L}$$

□

Orthogonality

Definition

We say two integrable functions f and g are orthogonal on an interval $[a, b]$ if

$$\int_a^b f(x)g(x)\,dx = 0$$

More generally, we say functions $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ (finite or infinitely many) are orthogonal on $[a, b]$ if

$$\int_a^b \phi_i(x)\phi_j(x)\,dx = 0 \quad \text{whenever} \quad i \neq j$$

We have already seen orthogonality of Legendre function.
We will study Fourier series w.r.t. different orthogonal systems.

Exercise

Consider the eigenfunctions

1. $\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \sin\frac{n\pi x}{L}, \ldots$
2. $1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \ldots$
3. $\sin\frac{\pi x}{2L}, \sin\frac{3\pi x}{2L}, \ldots, \sin\frac{(2n-1)\pi x}{2L}, \ldots$
4. $\cos\frac{\pi x}{2L}, \cos\frac{3\pi x}{2L}, \ldots, \cos\frac{(2n-1)\pi x}{2L}, \ldots$
5. $1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}, \ldots$

Show directly that eigenfunctions of (1-4) are orthogonal on $[0, L]$ and of (5) is orthogonal on $[-L, L]$.

We will study series expansions in terms of eigenfunctions. It is used to solve PDEs.

For this we consider the vector space of functions on $[a, b]$ and define an inner product on it

$$\langle f, g \rangle := \int_a^b f(x)g(x)dx$$

Denote by $L^2[a, b]$ the subspace of those functions satisfying $\langle f, f \rangle < \infty$.

To say this is a subspace, one needs to check that if $f, g \in L^2[a, b]$ then $f + g \in L^2[a, b]$. We shall assume this fact.

From now on, we will always be working with functions in some inner product space of the type $L^2[a, b]$. In such a space, the norm of a function is defined to be $\|f\| := \langle f, f \rangle^{1/2}$.

Theorem

Let $f \in L^2[-L, L]$. Then f can be written as a series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

which is called the Fourier series of f on $[-L, L]$. Here

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \qquad \text{and for } n > 0$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

The above series converges to f in norm, that is,

$$\lim_{N \longrightarrow \infty} \left\| f - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right\| = 0$$

We remark that the formula for the coefficients a_m's can be obtained by integrating $f(x)$ with $\cos\frac{m\pi x}{L}$ on $[-L, L]$, and using the facts that (1) we can exchange the integral and the sum, and (2) orthogonality of the different eigenfunctions.

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L}\cos\frac{m\pi x}{L}a_0 +$$

$$+\int_{-L}^{L}\cos\frac{m\pi x}{L}\sum_{n=1}^{\infty}\Big(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\Big)$$

$$= \int_{-L}^{L}\cos\frac{m\pi x}{L}a_0 + \sum_{n=1}^{\infty}a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} +$$

$$b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}$$

$$= a_m\int_{-L}^{L}\cos^2\frac{m\pi x}{L}dx$$

Qn. What about the convergence of series to $f(x)$?

Definition

A function f is said to be piecewise smooth if

1. f has atmost finitely many points of discontinuity.
2. f' exists and is continuous except at finitely many points.
3. $f(x_0+) = \lim_{x \to x_0^+} f(x)$ and $f'(x_0+) = \lim_{x \to x_0^+} f'(x)$ exists if $a \leq x_0 < b$.
4. $f(x_0-) = \lim_{x \to x_0^-} f(x)$ and $f'(x_0-) = \lim_{x \to x_0^-} f'(x)$ exists if $a < x_0 \leq b$.

Hence f is piecewise smooth if and only if
f, f' have atmost finitely many jump discontinuity.

Theorem

Let $f(x)$ be a piecewise smooth on $[-L, L]$.
Extend it to all of $\mathbb{R}$ by defining it periodically, that is, $f(x + 2L) = f(x)$.
Then the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

of f converges to

$$\frac{1}{2}[f(x^+) + f(x^-)]$$

at every point $x \in \mathbb{R}$.
Therefore, at every point x of continuity of f, the Fourier series converges to $f(x)$.

If we re-define $f(x)$ at every point of discontinuity x as

$$\frac{1}{2}[f(x^+) + f(x^-)]$$

then the Fourier series represents the function everywhere.
Thus two functions can have same Fourier series.

Let us now consider a function f such that f has only jump discontinuities, and if x is a such a point of jump discontinuity then $f(x) = \frac{f(x^+)+f(x^-)}{2}$.

The previous theorem tells us that the Fourier series converges to $f(x)$ for all $x \in [-L, L]$,
we may be tempted to infer that the error

$$E_N(x) = \left| F(x) - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right|$$

can be made as small as we want, for all $x \in [-L, L]$ by choosing N sufficiently large.

However this is NOT true if

- f is discontinuous at some point $\alpha \in (-L, L)$ or
- $f(-L+) \neq f(L-)$

The next result explains this.

- If f has a jump discontinuity at $\alpha \in (-L, L)$, then there exists sequence of points $u_N \in (-L, \alpha)$ and $v_N \in (\alpha, L)$ s.t.

$$\lim_{N\to\infty} u_N = \alpha, \quad E_N(u_N) \simeq .09\,|f(\alpha-) - f(\alpha+)|$$

$$\lim_{N\to\infty} v_N = \alpha, \quad E_N(v_N) \simeq .09\,|f(\alpha-) - f(\alpha+)|$$

Maximum of error $E_N(x) \not\to 0$ near α as $N \to \infty$.

- If $f(-L+) \neq f(L-)$, there exists u_N and v_N in $(-L, L)$ s.t.

$$\lim_{N\to\infty} u_N = -L, \quad E_N(u_N) \simeq .09\,|f(-L+) - f(L-)|$$

$$\lim_{N\to\infty} v_N = \alpha = L, \quad E_N(v_N) \simeq .09\,|f(-L+) - f(L-)|$$

This is called Gibbs phenomenon.

Example

Let us find the Fourier series of the piecewise smooth function

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

on $[-2, 2]$.

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\left[\int_{-2}^{0}(-x)\,dx + \int_0^2 \frac{1}{2}\,dx\right] = \frac{3}{4}$$

If $n \geq 1$, then

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx \\ &= \frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx + \int_0^2 \frac{1}{2}\cos\frac{n\pi x}{2}\,dx\right] \end{aligned}$$

Example (continued ...)

$$= \frac{1}{2}\left[-x\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \int_{-2}^{0}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx + 0\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}\left(-\cos\frac{n\pi x}{2}\right)\Big|_{-2}^{0}$$

$$= \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(-x)\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2}\frac{1}{2}\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2n\pi}(1 + 3\cos n\pi)$$

Example (continued ...)

Thus, the Fourier series of $f(x)$ is

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi - 1}{n^2}\cos\frac{n\pi x}{2} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1+3\cos n\pi}{n}\sin\frac{n\pi x}{2}$$

□

Let us compute $F(x)$ at discontinuous points.

Example (continued ...)

$$F(-2) = F(2) = \frac{1}{2}\left(f(-2+) + f(2-)\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$$

$$F(0) = \frac{1}{2}\left(f(0-) + f(0+)\right) = \frac{1}{2}\left(0 + \frac{1}{2}\right) = \frac{1}{4}$$

To summarize,

$$F(x) = \begin{cases} 5/4, & x = \pm 2 \\ -x, & -2 < x < 0 \\ 1/4, & x = 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

- EVP 1. $y'' + \lambda y = 0, \;\; y(0) = 0, \; y(L) = 0$

has infinitely many positive eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ for $n \geq 1$ with associated eigenfunctions

$$y_n = \sin \frac{n\pi x}{L}.$$

- EVP 2. $y'' + \lambda y = 0, \;\; y'(0) = 0, \; y'(L) = 0$

has eigenvalue $\lambda_0 = 0$ with eigenfunction $y_0 = 1$.
has infinitely many positive eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ for $n \geq 1$ with associated eigenfunctions

$$y_n = \cos \frac{n\pi x}{L}.$$

• EVP 3. $y'' + \lambda y = 0$, $y(0) = 0$, $y'(L) = 0$
has infinitely many positive eigenvalues
$\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$, $n = 1, 2, \ldots$
with associated eigenfunctions

$$y_n = \sin \frac{(2n-1)\pi x}{2L}.$$

• EVP 4. $y'' + \lambda y = 0$, $y'(0) = 0$, $y(L) = 0$
has infinitely many positive eigenvalues
$\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$, $n = 1, 2, \ldots$
with associated eigenfunctions

$$y_n = \cos \frac{(2n-1)\pi x}{2L}.$$

• EVP 5. $y'' + \lambda y = 0,\ \ y(-L) = y(L),\ y'(-L) = y'(L)$
has an eigenvalue $\lambda_0 = 0$ with eigenfunction $y_0 = 1$
and infinitely many positive eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$
with associated eigenfunctions

$$y_{1n} = \cos\frac{n\pi x}{L} \quad \text{and} \quad y_{2n} = \sin\frac{n\pi x}{L}.$$

• Eigenfunctions of EVP (1-4) are orthogonal on $[0, L]$ w.r.t. inner product $\langle f, g\rangle = \displaystyle\int_0^L f(x)g(x)dx$
• Eigenfunctions of EVP 5 is orthogonal on $[-L, L]$ w.r.t. inner product $\langle f, g\rangle = \displaystyle\int_{-L}^L f(x)g(x)dx$.

Fourier Series.

Let $f \in L^2([-L,L])$ be piecewise smooth. Extend f to $\mathbb{R}$ as a periodic function of period $2L$.

The Fourier series of f is

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx, \quad n > 0$$

- $F(x) = \frac{1}{2}[f(x^+) + f(x^-)]$ for all $x \in \mathbb{R}$.

Fourier sine series

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L}$$

To see this, let us first extend f to $[-L, L]$ by defining $f(x) = -f(-x)$ for $x \in [-L, 0)$. Denote the extension by $\tilde{f}$.

Then we know that $\tilde{f}$ has a Fourier expansion

$$\tilde{f}(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \tilde{f}(x) dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \cos \frac{n\pi x}{L} dx \qquad n > 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \sin \frac{n\pi x}{L} dx$$

Now note that by the way $\tilde{f}$ has been defined, it is an odd function. Thus, $a_0 = 0$.

Since $\cos \frac{n\pi x}{L}$ is an even function and $\tilde{f}$ is odd, it follows $\tilde{f}(x) \cos \frac{n\pi x}{L}$ is an odd function. Thus, $a_n = 0$.

This proves that

$$\tilde{f}(x) = \sum_{n \geq 1} a_n \sin \frac{n\pi x}{L}$$

Restricting this expansion to $[0, L]$ we get the required expansion of f.

Fourier cosine series

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{L}$$

To see this, let us first extend f to $[-L, L]$ by defining $f(x) = f(-x)$ for $x \in [-L, 0)$. Denote the extension by $\tilde{f}$.

Then we know that $\tilde{f}$ has a Fourier expansion

$$\tilde{f}(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \tilde{f}(x) dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \cos \frac{n\pi x}{L} dx \qquad n > 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \sin \frac{n\pi x}{L} dx$$

Now note that by the way $\tilde{f}$ has been defined, it is an even function.

Since $\sin \frac{n\pi x}{L}$ is an odd function and $\tilde{f}$ is even, it follows $\tilde{f}(x) \sin \frac{n\pi x}{L}$ is an odd function. Thus, $b_n = 0$.

This proves that

$$\tilde{f}(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{L}$$

Restricting this expansion to $[0, L]$ we get the required expansion of f.

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = \sum_{n \geq 1} a_n \sin \frac{(2n-1)\pi x}{2L}$$

Let $f \in L^2([0, L])$. Extend f to f_1 on $[0, 2L]$ as $f_1(x) = f(2L - x)$ for $x \in (L, 2L)$.

Fourier sine series of f_1 on $[0, 2L]$ is

$$F(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{2L}$$

$$b_n = \frac{2}{2L} \int_0^{2L} f_1(x) \sin \frac{n\pi x}{2L}\, dx$$

$$= \frac{1}{L} \int_0^{L} f(x) \sin \frac{n\pi x}{2L} dx + \frac{1}{L} \int_L^{2L} f(2L - x) \sin \frac{n\pi x}{2L} dx$$

Expansion in terms of eigenfunctions of EVP3

$$\int_L^{2L} f(2L-x)\sin\frac{n\pi x}{2L}dx$$

$$(x'=2L-x), \qquad = \int_L^0 f(x')\sin(n\pi - \frac{n\pi x'}{2L})(-dx')$$

$$\int_0^L (-1)^{n+1} f(x)\sin\frac{n\pi x}{2L}\,dx$$

$$b_n = \frac{1}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}dx + \frac{1}{L}\int_0^L (-1)^{n+1} f(x)\sin\frac{n\pi x}{2L}dx$$

So $b_{2n}=0$, $b_{2n-1}=\dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx$.

Thus $F(x)=\displaystyle\sum_{n\geq 1} b_{2n-1}\sin\frac{(2n-1)\pi x}{2L}$.

Expansion in terms of eigenfunctions of EVP3

The Mixed Fourier sine series of $f \in L^2([0, L])$ is the restriction of Fourier sine series of f_1 to $[0, L]$, i.e.

$$F(x) = \sum_{n \geq 1} c_n \sin \frac{(2n-1)\pi x}{2L}$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} \, dx$$

This is the Fourier series of f on $[0, L]$ w.r.t. orthogonal system of eigenfunctions

$$B = \{\sin \frac{\pi x}{2L}, \sin \frac{3\pi x}{2L}, \ldots, \sin \frac{(2n-1)\pi x}{2L}, \ldots\}$$

of EVP 3 : $\boxed{y'' + \lambda y = 0, \quad y(0) = 0 = y'(L)}$.

Mixed Fourier cosine series

Let $f \in L^2([0, L])$. Extend f to f_1 on $[0, 2L]$ as $f_1(x) = -f(2L - x)$ for $x \in (L, 2L)$.

Fourier cosine series of f_1 on $[0, 2L]$ is

$$F(x) = \sum_{n=1}^{\infty} d_n \cos \frac{(2n-1)\pi x}{2L}, d_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

This is the Fourier series of f on $[0, L]$ w.r.t. orthogonal system of eigenfunctions

$$B = \{\cos \frac{\pi x}{2L}, \cos \frac{3\pi x}{2L}, \ldots, \cos \frac{(2n-1)\pi x}{2L}, \ldots\}$$

of EVP 4 : $\boxed{y'' + \lambda y = 0, \; y'(0) = 0 = y(L)}$.

A useful observation

Often we need to find Fourier expansion of polynomial functions in terms of the eigenfunctions of Problems 1-4 satisfying the boundary conditions.

We can use "derivative transfer principle" to find Fourier coefficients.

In EVP 1 with $f(0) = 0 = f(L)$, we get Fourier sine series on $[0, L]$.

$$F(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{n\pi} \int_0^L f'(x) \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{-2}{L} \left(\frac{L}{n\pi} \right)^2 \int_0^L f''(x) \sin \frac{n\pi x}{L} \, dx$$

In EVP (2) with $f'(0) = 0 = f'(L)$, we get Fourier cosine series on $[0, L]$, where for $n \geq 1$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = \frac{-2}{n\pi} \int_0^L f'(x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{-2L}{n^2\pi^2} \int_0^L f''(x) \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \left(\frac{L}{n\pi} \right)^3 \int_0^L f'''(x) \sin \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L} \int_0^L f(x) \, dx$$

In EVP 3 with $f(0) = 0 = f'(L)$, we get Mixed Fourier sine series on $[0, L]$.

$$F(x) = \sum_{n\geq 1} c_n \sin \frac{(2n-1)\pi x}{2L}\, dx$$

$$c_n = \frac{2}{L}\int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{4}{(2n-1)\pi}\int_0^L f'(x) \cos \frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

In EVP 4 with $f'(0) = 0 = f(L)$, we get Mixed Fourier cosine series on $[0, L]$.

$$F(x) = \sum_{n \geq 1} d_n \cos \frac{(2n-1)\pi x}{2L} \, dx$$

$$d_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

$$= \frac{-4}{(2n-1)\pi} \int_0^L f'(x) \sin \frac{(2n-1)\pi x}{2L} \, dx$$

$$= \frac{-2}{L} \left(\frac{2L}{(2n-1)\pi} \right)^2 \int_0^L f''(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

Example. Find the Fourier sine expansion of

$$f(x) = x(x^2 - 3Lx + 2L^2) \quad \text{on} \quad [0, L]$$

Note $f(0) = 0 = f(L)$, $f''(x) = 6(x - L)$, Fourier sine coefficient

$$b_n = \frac{-2}{L}\left(\frac{L}{n\pi}\right)^2 \int_0^L f''(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{-12L}{n^2\pi^2} \int_0^L (x - L) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{12L^2}{n^3\pi^3}\left[(x - L)\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}\, dx\right]$$

$$= \frac{12L^2}{n^3\pi^3}\left[L - \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L\right] = \frac{12L^3}{n^3\pi^3}$$

Therefore, the Fourier sine expansion of $f(x)$ on $[0, L]$ is

$$\frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{L}$$ □

Example. Find the Fourier cosine expansion of

$$f(x) = x^2(3L - 2x) \quad \text{on} \quad [0, L]$$

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L (3Lx^2 - 2x^3)\,dx \\ &= \frac{1}{L}\left(Lx^3 - \frac{x^4}{2}\right)_0^L \\ &= \frac{L^3}{2} \end{aligned}$$

$$f'(x) = 6Lx - 6x^2 \implies f'(0) = f'(L) = 0$$

Note $f'''(x) = -12$. We get

$$a_n = \frac{2}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^L f'''(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{-24}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^L \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{24}{L}\left(\frac{L}{n\pi}\right)^4 \cos\frac{n\pi x}{L}\Big|_0^L = \frac{24L^3}{n^4\pi^4}\ [(-1)^n - 1]$$

Thus $a_{2n} = 0$ and $a_{2n-1} = \dfrac{-48L^3}{(2n-1)^4\pi^4}$.

Thus Fourier cosine expansion of $f(x)$ on $[0, L]$ is

$$\frac{L^3}{2} - \frac{48L^3}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\frac{(2n-1)\pi x}{L}$$

Example Find the mixed Fourier sine expansion of

$$f(x) = x(2x^2 - 9Lx + 12L^2) \quad \text{on} \quad [0, L]$$

Since $f(0) = 0 = f'(L)$ and $f''(x) = 6(2x - 3L)$, we get

$$c_n = \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-48L}{(2n-1)^2\pi^2} \int_0^L (2x - 3L) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{96L^2}{(2n-1)^3\pi^3}\Big[(2x - 3L) \cos \frac{(2n-1)\pi x}{2L}\Big|_0^L$$

$$-2\int_0^L \cos \frac{(2n-1)\pi x}{2L}\, dx\Big]$$

$$= \frac{96L^2}{(2n-1)^3\pi^3}\left[3L - \frac{4L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$
$$= \frac{96L^3}{(2n-1)^3\pi^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]$$

Therefore, the mixed Fourier sine expansion of $f(x)$ on $[0, L]$ is

$$c\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$$

with $c = \dfrac{96L^3}{\pi^3}$.

Example. Find the mixed Fourier cosine expansion of $f(x) = 3x^3 - 4Lx^2 + L^3$ on $[0, L]$

Soln. $f'(0) = 0 = f(L)$ $f''(x) = 2(9x - 4L)$, we get

$$d_n = \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x) \cos\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{-16L}{(2n-1)^2\pi^2}\int_0^L (9x - 4L)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\Big[(9x-4L)\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L$$

$$-9\int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx\Big]$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\left[(-1)^{n+1}5L + \frac{18L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$

$$= \frac{32L^3}{(2n-1)^3\pi^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]$$

Therefore, the Mixed Fourier cosine expansion of $f(x)$ on $[0, L]$ is

$$\frac{32L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$$

$\square$