MA-207 Differential Equations II

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The boundary condition in problem 5 is called periodic.

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Problems $1 - 5$ are called eigenvalue problems. Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions.

Theorem

1. *Problems* $1-5$ *have no negative eigenvalues.*
2. $\lambda = 0$ *is an eigenvalue of Problems* 2 *and* 5 *with associated eigenfunctions* $y_0 = 1$.
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Rewrite the differential equation as $y'' = a^2 y$. The general solution to this is $y(x) = Ce^{ax} + De^{-ax}$. In problem 1 we have the condition $y(0) = y(L) = 0$. This forces that $C + D = 0$ and $Ce^{aL} + De^{-aL} = 0$. One checks easily that this forces $C = D = 0$.

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In problem 2 we have the condition that $y'(0) = y'(L) = 0$. This gives $aC - aD = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Since $a \neq 0$, this forces $C = D = 0$.

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In problem 3 we have the conditions $y(0) = y'(L) = 0$. This gives $C + D = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Again this forces $C = D = 0$.

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Similarly, do the other problems.

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In problem 5, we have $y(-L) = y(L)$, that is, $-aL + b = aL + b$. This forces that $a = 0$. Thus, in this case too $y(x) = \text{constant}$. □

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The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$y_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots .$$

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We will study Fourier series w.r.t. different orthogonal systems.

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Show directly that eigenfunctions of (1-4) are orthogonal on $[0, L]$

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Show directly that eigenfunctions of (1-4) are orthogonal on $[0, L]$ and of (5) is orthogonal on $[-L, L]$.

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From now on, we will always be working with functions in some inner product space of the type $L^2[a, b]$. In such a space, the norm of a function is defined to be $\|f\| := \langle f, f\rangle^{1/2}$.

Fourier Series

Theorem

Let $f \in L^2[-L, L]$*. Then* f *can be written as a series*

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

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$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx \qquad \text{and for } n > 0$$

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The above series converges to f in norm, that is,

$$\lim_{N \longrightarrow \infty} \left\| f - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right\| = 0$$

We remark that the formula for the coefficients a_m's can be obtained by integrating $f(x)$ with $\cos \frac{m\pi x}{L}$ on $[-L, L]$, and using the facts that (1) we can exchange the integral and the sum, and (2) orthogonality of the different eigenfunctions.

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$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L}\cos\frac{m\pi x}{L}a_0 +$$

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$$=\int_{-L}^{L}\cos\frac{m\pi x}{L}a_0+\sum_{n=1}^{\infty}a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}+$$

$$b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}$$

$$=a_m\int_{-L}^{L}\cos^2\frac{m\pi x}{L}dx$$

Convergence of Fourier series

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Hence f is piecewise smooth if and only if
f, f' have atmost finitely many jump discontinuity.

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then the Fourier series represents the function everywhere.
Thus two functions can have same Fourier series.

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$$E_N(x) = \left| F(x) - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right|$$

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can be made as small as we want, for all $x \in [-L, L]$ by choosing N sufficiently large.

However this is NOT true if

- f is discontinuous at some point $\alpha \in (-L, L)$ or
- $f(-L+) \neq f(L-)$

The next result explains this.

- If f has a jump discontinuity at $\alpha \in (-L, L)$, then there exists sequence of points $u_N \in (-L, \alpha)$ and $v_N \in (\alpha, L)$ s.t.

$$\lim_{N\to\infty} u_N = \alpha, \quad E_N(u_N) \simeq .09\,|f(\alpha-) - f(\alpha+)|$$

$$\lim_{N\to\infty} v_N = \alpha, \quad E_N(v_N) \simeq .09\,|f(\alpha-) - f(\alpha+)|$$

Maximum of error $E_N(x) \not\to 0$ near α as $N \to \infty$.

- If $f(-L+) \neq f(L-)$, there exists u_N and v_N in $(-L, L)$ s.t.

$$\lim_{N\to\infty} u_N = -L, \quad E_N(u_N) \simeq .09\,|f(-L+) - f(L-)|$$

$$\lim_{N\to\infty} v_N = \alpha = L, \quad E_N(v_N) \simeq .09\,|f(-L+) - f(L-)|$$

This is called Gibbs phenomenon.

Example

Let us find the Fourier series of the piecewise smooth function

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

on $[-2, 2]$.

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If $n \geq 1$, then

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx \\ &= \frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2}\frac{1}{2}\cos\frac{n\pi x}{2}\,dx\right] \end{aligned}$$

Example (continued ...)

$$= \frac{1}{2}\left[-x\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \int_{-2}^{0}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx + 0\right]$$

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$$= \frac{1}{2n\pi}(1 + 3\cos n\pi)$$

Example (continued ...)

Thus, the Fourier series of $f(x)$ is

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi - 1}{n^2}\cos\frac{n\pi x}{2} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1+3\cos n\pi}{n}\sin\frac{n\pi x}{2}$$

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To summarize,

$$F(x) = \begin{cases} 5/4, & x = \pm 2 \\ -x, & -2 < x < 0 \\ 1/4, & x = 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

• EVP 1. $y'' + \lambda y = 0, \;\; y(0) = 0, \, y(L) = 0$

has infinitely many positive eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ for $n \geq 1$ with associated eigenfunctions

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• EVP 3. $y'' + \lambda y = 0$, $y(0) = 0$, $y'(L) = 0$
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$\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$, $n = 1, 2, \ldots$
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Restricting this expansion to $[0, L]$ we get the required expansion of f.

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Expansion in terms of eigenfunctions of EVP3

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Fourier sine series of f_1 on $[0, 2L]$ is

$$F(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{2L}$$

$$b_n = \frac{2}{2L} \int_0^{2L} f_1(x) \sin \frac{n\pi x}{2L} \, dx$$

$$= \frac{1}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L} dx + \frac{1}{L} \int_L^{2L} f(2L - x) \sin \frac{n\pi x}{2L} dx$$

Expansion in terms of eigenfunctions of EVP3

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So $b_{2n}=0$, $b_{2n-1} = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx$.

Thus $F(x) = \displaystyle\sum_{n\geq 1} b_{2n-1}\sin\frac{(2n-1)\pi x}{2L}$.

Expansion in terms of eigenfunctions of EVP3

The Mixed Fourier sine series of $f \in L^2([0,L])$ is the restriction of Fourier sine series of f_1 to $[0,L]$, i.e.

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This is the Fourier series of f on $[0, L]$ w.r.t. orthogonal system of eigenfunctions

$$B = \{\sin \frac{\pi x}{2L}, \sin \frac{3\pi x}{2L}, \ldots, \sin \frac{(2n-1)\pi x}{2L}, \ldots\}$$

of EVP 3 : $\boxed{y'' + \lambda y = 0, \quad y(0) = 0 = y'(L)}$.

Expansion in terms of eigenfunctions of EVP4

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Let $f \in L^2([0, L])$. Extend f to f_1 on $[0, 2L]$ as $f_1(x) = -f(2L - x)$ for $x \in (L, 2L)$.

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A useful observation

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We can use "derivative transfer principle" to find Fourier coefficients.

In EVP 1 with $f(0) = 0 = f(L)$, we get Fourier sine series on $[0, L]$.

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In EVP (2) with $f'(0) = 0 = f'(L)$, we get Fourier cosine series on $[0, L]$, where for $n \geq 1$,

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$$= \frac{12L^2}{n^3\pi^3}\left[(x - L)\cos \frac{n\pi x}{L}\bigg|_0^L - \int_0^L \cos \frac{n\pi x}{L}\, dx\right]$$

$$= \frac{12L^2}{n^3\pi^3}\left[L - \frac{L}{n\pi}\sin \frac{n\pi x}{L}\bigg|_0^L\right] = \frac{12L^3}{n^3\pi^3}$$

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$$f(x) = x^2(3L - 2x) \quad \text{on} \quad [0, L]$$

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Note $f'''(x) = -12$. We get

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Example Find the mixed Fourier sine expansion of

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Therefore, the mixed Fourier sine expansion of $f(x)$ on $[0, L]$ is

$$c\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$$

with $c = \dfrac{96L^3}{\pi^3}$.

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$$-9\int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx\Big]$$

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Therefore, the Mixed Fourier cosine expansion of $f(x)$ on $[0, L]$ is

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□