MA-207 Differential Equations II

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Now we will start the study of Partial differential equations.

A partial differential equation (PDE) is an equation for an unknown function u that involves independent variables $x, y, \dots$, the function u and the partial derivatives of u.
The order of the PDE is the order of the highest partial derivative of u in the equation.
Examples of some famous PDEs.

1. $u_t - k(u_{xx} + u_{yy}) = 0$ two dimensional Heat equation, order 2.
2. $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$ two dimensional wave equation, order 2.
3. $u_{xx} + u_{yy} = 0$ two dimensional Laplace equation, order 2.
4. $u_{tt} + u_{xxxx}$ Beam equation, order 4.

Examples of non-famous PDE's (I made it up).

1. $u_x + \sin(u_y) = 0$, order 1.
2. $3x^2 \sin(xy) e^{-xy^2} u_{xx} + \log(x^2 + y^2) u_y = 0$, order 2.

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A PDE is said to be "linear" if it is linear in u and its partial derivatives i.e. it is a degree 1 polynomial in u and its partial derivatives.

Heat equation, Wave equation, Laplace equation and Beam equation are linear PDEs.

In the above two non-famous examples, the first is non-linear while the second is linear.

The general form of first order linear PDE in two variables x, y is

$$A(x,y)u_x + B(x,y)u_y + C(x,y)u = f(x,y)$$

The general form of first order linear PDE in three variables x, y, z is

$$Au_x + Bu_y + Cu_z + Du = f$$

where coefficients A, B, C, D and f are functions of x, y and z. The general form of second order linear PDE in two variables x, y is

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f$$

where coefficients A, B, C, D, E, F and f are functions of x and y. When $A \ldots, F$ are all constants, then its a linear PDE with constant coefficients.

Linear Partial Differential Operator

Second order linear PDE in two variable can be written as $Lu = f$, where

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

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is the linear differential operator. It is called linear since the map $u \mapsto Lu$ is a linear map.

Examples. Laplace operator in $\mathbb{R}^2$ is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Heat and Wave operator in one space variable are

$$H = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \qquad \Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

Classification of second order linear PDE

Consider the linear differential operator L in $\mathbb{R}^2$.

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

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where $A, \ldots, F$ are functions of x and y.
To the operator L, we associate the discriminant $\mathbb{D}(x, y)$ given by

$$\mathbb{D}(x, y) = A(x, y)C(x, y) - B^2(x, y)$$

The operator L or the PDE $Lu = f$ is said to be

- elliptic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) > 0$,
- hyperbolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) < 0$,
- parabolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) = 0$.

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- Two dimensional Laplace operator Δ is elliptic in $\mathbb{R}^2$, since $\mathbb{D} = 1$.
- One dimensional Heat operator H is parabolic in $\mathbb{R}^2$, since $\mathbb{D} = 0$.
- One dimensional Wave operator $\Box$ is hyperbolic in $\mathbb{R}^2$, since $\mathbb{D} = -1$.

When the coefficients of an operator L are not constant, the type may vary from point to point.

Example. Consider the Tricomi operator (well known)

$$T = \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}$$

The discriminant $\mathbb{D} = x$.
Hence T is elliptic in the half-plane $x > 0$,
hyperbolic in the half-plane $x < 0$ and
parabolic on the y-axis.

Remark about terminology

Consider

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

at the point (x_0, y_0). If we replace $\partial/\partial x$ by ξ and $\partial/\partial y$ by η and evaluate $A, \ldots, F$ at (x_0, y_0), then L becomes a polynomial in 2 variables

$$P(\xi, \eta) = A\xi^2 + 2B\xi\eta + C\eta^2 + D\xi + E\eta + F$$

Consider the curves in (ξ, η)-plane given by

$$P(\xi, \eta) = \text{constant}$$

then these curves are elliptic if $\mathbb{D}(x_0, y_0) > 0$, hyperbolic if $\mathbb{D}(x_0, y_0) < 0$ and parabolic if $\mathbb{D}(x_0, y_0) = 0$.

Second order linear operators in $\mathbb{R}^3$

The classification is done analogously by associating a polynomial of degree 2 in three variables to L and considering the surfaces defined by level sets of the polynomial.
These surfaces are either ellipsoids, hyperboloids, or paraboloids.
The operator L is accordingly labeled as elliptic, hyperbolic or parabolic.
We can also proceed as follows; Consider

$$L = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + 2c\frac{\partial^2}{\partial x \partial z} + d\frac{\partial^2}{\partial y^2} + 2e\frac{\partial^2}{\partial y \partial z} + f\frac{\partial^2}{\partial z^2}$$
$$+ \text{ lower order terms}$$

where $a, b, \ldots$ are functions of (x, y, z).

To L, we associate the symmetric matrix

$$M(x,y,z) = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Here the (i,j)-th entry is the coefficient of $\dfrac{\partial^2}{\partial x_i \partial x_j}$.
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- L is elliptic at (x_0, y_0, z_0) if all three eigen values of $M(x_0, y_0, z_0)$ are of same sign.
- L is hyperbolic at (x_0, y_0, z_0) if two eigen values are of same sign and one of different sign.
- L is parabolic at (x_0, y_0, z_0) if one of the eigenvalue is zero.

Principle of superposition

Let L be a linear differential operator.
The PDE $Lu = 0$ is called homogeneous and
the PDE $Lu = f$, $(f \neq 0)$ is non-homogeneous.

Principle 1. If $u_1, \dots, u_N$ are solutions of $Lu = 0$ and $c_1, \dots, c_N$ are constants, then $\displaystyle\sum_{i=1}^{N} c_i u_i$ is also a solution of $Lu = 0$.

In general, space of solutions of $Lu = 0$ contains infinitely many independent solutions and we may need to use infinite linear combinations of them.

Principle 2.

Assume

- $u_1, u_2, \ldots$ are infinitely many solutions of $Lu = 0$.
- the series $w = \sum_{i \geq 1} c_i u_i$ with $c_1, c_2, \ldots$ constants, converges to a

twice differentiable function;

- term by term partial differentiation is valid for the series, i.e.

$Dw = \sum_{i \geq 1} c_i D u_i$, D is any partial differentiation of order 1 or 2.

Then w is again a solution of $Lu = 0$.

Principle 3 for non-homogeneous PDE.

If u_i is a solution of $Lu = f_i$, then

$$w = \sum_{i=1}^{N} c_i u_i$$

with constants c_i, is a solution of $Lu = \sum_{i=1}^{N} c_i f_i$.

One-dimensional heat equation

The temperature evolution of a thin rod of length L is decribed by the PDE

$$u_t = k^2 u_{xx}, \quad 0 < x < L,\ t > 0,$$

called one-dimensional heat equation.
Here k is a positive constant.

x is the space variable and t is the time variable.

$u(x,t)$ is the temperature at point x and time t.

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We must also specify boundary conditions that u must satisfy at the two endpoints of the rod for all $t > 0$.

We call this problem an initial-boundary value problem IBVP.

We consider different kinds of boundary conditions.

In each case, we use method of separation of variables.
Suppose

$$v(x,t) = X(x)\,T(t)$$

Substituting this in the Heat equation $\boxed{u_t = k^2 u_{xx}}$

$$T'(t)X(x) = k^2 X''(x)T(t).$$

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We can now separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{k^2 T(t)}$$

The equality is between a function of x and a function of t, so both must be constant, say $-\lambda$.
We need to solve

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T'(t) = -k^2 \lambda T(t).$$

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Initial-boundary value problem is

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(L,t) = 0, && t > 0 \\ &u(x,0) = f(x), && 0 \leq x \leq L \end{aligned}$$

The endpoints of the rod are maintained at temperature 0 at all time t.

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The endpoints of the rod are maintained at temperature 0 at all time t.

(The rod is isolated from the surroundings except at the endpoints from where heat will be lost to the surrounding.)

Assuming the solution in the form $\boxed{v(x,t) = X(x)T(t)}$

$$v(0,t) = X(0)T(t) = 0 \quad \text{and} \quad v(L,t) = X(L)T(t) = 0$$

we don't want T to be identically zero, we get

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

We need to solve eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(L) = 0, \qquad (*)$$

and $$T'(t) = -k^2 \lambda T(t) \implies T(t) = exp(-k^2 \lambda t)$$

The eigenvalues of $(*)$ are

$$\lambda_n = \frac{n^2 \pi^2}{L^2}$$

with associated eigenfunctions

$$X_n = \sin \frac{n\pi x}{L}, \quad n \geq 1.$$

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We get infinitely many solutions for IBVP, one for each $n \geq 1$

$$\begin{aligned} v_n(x,t) &= T_n(t) X_n(x) \\ &= exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \sin \frac{n\pi x}{L} \end{aligned}$$

Note $$v_n(x,0) = \sin \frac{n\pi x}{L}$$

Therefore

$$v_n(x,t) = exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\sin\frac{n\pi x}{L}$$

satisfies the IBVP

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More generally, if $\alpha_1, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m}\alpha_n\, exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with initial condition

$$u_m(x,0) = \sum_{n=1}^{m}\alpha_n\sin\frac{n\pi x}{L}.$$

Let us consider the formal series

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \sin\frac{n\pi x}{L}$$

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Setting $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

To solve our IBVP we would like to have

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L} \qquad 0 \leq x \leq L$$

Is it possible that f has such an expansion?

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Is it possible that f has such an expansion?

Given f on $[0, L]$, it has a Fourier sine series

$$f(x) = \sum_{n\geq 1} b_n \sin\frac{n\pi x}{L}$$

Definition

The formal solution of IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 && t > 0 \\ u(L,t) &= 0 && t > 0 \\ u(x,0) &= f(x) && 0 \le x \le L \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t \right) \sin \frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$ i.e.

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

$$u(x,t)=\sum_{n=1}^{\infty}\alpha_n\, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\sin\frac{n\pi x}{L}$$

We say $u(x,t)$ is a formal solution, since the series for $u(x,t)$ may NOT satisfy all the requirements of IBVP.

When it does, we say it is an actual solution of IBVP.

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Because of negative exponential in $u(x,t)$, the series in $u(x,t)$ converges for all $t > 0$.

Each term in $u(x,t)$ satisfies the heat equation and boundary condition.

If u_t and u_{xx} can be obtained by differentiating the series term by term, once w.r.t. t and twice w.r.t. x for $t > 0$, then u also satisfies these properties.

If $f(x)$ is continuous and piecewise smooth on $[0, L]$, then we can do it. Hence we get next result.

Theorem

$f(x)$: *continuous and piecewise smooth on* $[0, L]$
$f(0) = f(L) = 0$

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L} \quad \textit{with} \quad \alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

is Fourier sine series of f *on* $[0, L]$*. Then the IBVP*

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(L,t) = 0 && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

has a solution

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \sin \frac{n\pi x}{L}$$

Here u_t *and* u_{xx} *can be obtained by term-wise differentiation for* $t > 0$.

Example

Let $f(x) = x(x^2 - 3Lx + 2L^2)$. Solve IBVP

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The Fourier sine expansion of $f(x)$ is

$$S(x) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

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Therefore, the solution of IBVP is

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☐

Neumann boundary conditions

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Assuming the solution in the form $\boxed{v(x,t) = X(x)T(t)}$

$$v_x(0,t) = X'(0)T(t) = 0 \quad \text{and} \quad v_x(L,t) = X'(L)T(t) = 0$$

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and $\qquad T'(t) = -k^2 \lambda T(t) \implies T(t) = exp(-k^2 \lambda t)$

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We get infinitely many solutions for IBVP, one for each $n \geq 0$

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$$X_n = \cos\frac{n\pi x}{L}, \quad n \geq 0.$$

We get infinitely many solutions for IBVP, one for each $n \geq 0$

$$\begin{aligned} v_n(x,t) &= T_n(t)X_n(x) \\ &= exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\cos\frac{n\pi x}{L} \end{aligned}$$

Note $\qquad v_n(x,0) = \cos\frac{n\pi x}{L}$

Therefore

$$v_n(x,t) = exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\cos\frac{n\pi x}{L}$$

satisfies the IBVP

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = 0 && t > 0 \\ &u_x(L,t) = 0 && t > 0 \\ &u(x,0) = \cos\frac{n\pi x}{L} && 0 \le x \le L \end{aligned}$$

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More generally, if $\alpha_0, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=0}^{m} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\cos\frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with initial condition

$$u_m(x,0) = \sum_{n=0}^{m} \alpha_n \cos\frac{n\pi x}{L}.$$

Let us consider the formal series

$$u(x,t)=\sum_{n=0}^{\infty}\alpha_n\, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\cos\frac{n\pi x}{L}$$

Let us consider the formal series

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n\, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right) \cos\frac{n\pi x}{L}$$

Setting $t = 0$ we get

$$u(x,0) = \sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{L}$$

To solve our IBVP we would like to have

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{L} \qquad 0 \leq x \leq L$$

Is it possible that f has such an expansion?

Let us consider the formal series

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Is it possible that f has such an expansion?

Given f on $[0,L]$, it has a Fourier cosine series

$$f(x)=\sum_{n\ge 0}a_n\cos\frac{n\pi x}{L}$$

Definition

The formal solution of IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u_x(0,t) &= 0 && t > 0 \\ u_x(L,t) &= 0 && t > 0 \\ u(x,0) &= f(x) && 0 \leq x \leq L \end{aligned}$$

is

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp \left(\frac{-n^2 \pi^2 k^2}{L^2} t \right) \cos \frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$ i.e.

$$\alpha_0 = \frac{1}{L} \int_0^L f(x)\, dx \qquad \alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx.$$

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right) \cos\frac{n\pi x}{L}$$

We say $u(x,t)$ is a formal solution, since the series for $u(x,t)$ may NOT satisfy all the requirements of IBVP.

When it does, we say it is an actual solution of IBVP.

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When it does, we say it is an actual solution of IBVP.

Because of negative exponential in $u(x,t)$, the series in $u(x,t)$ converges for all $t > 0$.

Each term in $u(x,t)$ satisfies the heat equation and boundary condition.

If u_t and u_{xx} can be obtained by differentiating the series term by term, once w.r.t. t and twice w.r.t. x for $t > 0$, then u also satisfies these properties.

If $f(x)$ is continuous and piecewise smooth on $[0, L]$, then we can do it. Hence we get next result.

Theorem

$f(x)$ is continuous, piecewise smooth on $[0, L]$; $f'(0) = f'(L) = 0$.

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{L} \quad \textit{with}$$

$$\alpha_0 = \frac{1}{L} \int_0^L f(x)\,dx \qquad \alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx$$

is Fourier sine series of f on $[0, L]$. Then the IBVP

$$\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
&u_x(0,t) = 0 && t > 0 \\
&u_x(L,t) = 0 && t > 0 \\
&u(x,0) = f(x) && 0 \le x \le L
\end{aligned}$$

has a solution

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right) \cos \frac{n\pi x}{L}$$

Here u_t and u_{xx} can be obtained by term-wise differentiation for $t > 0$.

Example

Let $f(x) = x$ on $[0, L]$. Solve IBVP

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = 0 && t > 0 \\ &u_x(L,t) = 0 && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

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The Fourier cosine expansion of $f(x)$ is

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}.$$

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Therefore, the solution of IBVP is

$u(x,t) =$

$$\frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} exp\left(\frac{-(2n-1)^2 \pi^2 k^2}{L^2} t \right) \cos \frac{(2n-1)n\pi x}{L}.$$

Definition (Formal solution for Dirichlet boundary)

The formal solution of IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 && t > 0 \\ u(L,t) &= 0 && t > 0 \\ u(x,0) &= f(x) && 0 \le x \le L \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t \right) \sin \frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$ i.e.

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

Definition (Formal solution for Neumann boundary condition)

The formal solution of IBVP

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = 0 && t > 0 \\ &u_x(L,t) = 0 && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

is

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \cos\frac{n\pi x}{L}$$

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Non homogeneous PDE: Dirichlet boundary condition

Let us now consider the following PDE

$$\begin{aligned}
&u_t - k^2 u_{xx} = F(x,t) && 0 < x < L, \quad t > 0\\
&u(0,t) = f_1(t) && t > 0\\
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\end{aligned}$$

How do we solve this?

Non homogeneous PDE: Dirichlet boundary condition

Let us now consider the following PDE

$$\begin{aligned} &u_t - k^2 u_{xx} = F(x,t) && 0 < x < L, \quad t > 0 \\ &u(0,t) = f_1(t) && t > 0 \\ &u(L,t) = f_2(t) && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

How do we solve this?

Let us first make the substitution

$$z(x,t) = u(x,t) - (1 - \frac{x}{L}) f_1(t) - \frac{x}{L} f_2(t)$$

Then clearly

- $z_t - k^2 z_{xx} = G(x,t)$
- $z(0,t) = 0$
- $z(L,t) = 0$
- $z(x,0) = g(x)$

Non homogeneous PDE: Dirichlet boundary condition

It is clear that we would have solved for u iff we have solved for z. In view of this observation, let us try and solve the problem for z.

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By observing the boundary conditions, we guess that we should try and look for a solution of the type

$$z(x,t) = \sum_{n \geq 1} Z_n(t) \sin(\frac{n\pi x}{L})$$

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Differentiating the above term by term we get that is satisfies the equation

$$z_t - k^2 z_{xx} = \sum_{n\geq 1} \big(Z_n'(t) + \frac{k^2n^2\pi^2}{L^2}Z_n(t)\big)\sin(\frac{n\pi x}{L})$$

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Let us write

$$G(x,t) = \sum_{n\geq 1} G_n(t)\sin(\frac{n\pi x}{L})$$

Non homogeneous PDE: Dirichlet boundary condition

Thus, if we need $z_t - k^2 z_{xx} = G(x,t)$ then we should have that

$$G_n(t) = Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t) \qquad (*)$$

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We also need that $z(x,0) = g(x)$.

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If

$$g(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L}$$

then we should have that

$$Z_n(0) = b_n \qquad (!)$$

Clearly, there is a unique solution to the differential equation $(*)$ with initial condition (!).

Non homogeneous PDE: Dirichlet boundary condition

The solution to the above equation is given by

$$Z_n(t) = Ce^{-\frac{k^2n^2\pi^2}{L^2}t} + e^{-\frac{k^2n^2\pi^2}{L^2}t}\int_0^t G_n(s)e^{\frac{k^2n^2\pi^2}{L^2}s}ds$$

We can find the constant using the initial condition.

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We can find the constant using the initial condition.

Thus, we let $Z_n(t)$ be this unique solution, then the series

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\frac{n\pi x}{L})$$

solves our non homogeneous PDE with Dirichlet boundary conditions for z.

Non homogeneous PDE: Dirichlet boundary condition

Example

Let us now consider the following PDE

$$\begin{aligned} &u_t - u_{xx} = e^t && 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(1,t) = 0 && t > 0 \\ &u(x,0) = x(x-1) && 0 \leq x \leq 1 \end{aligned}$$

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From the boundary conditions $u(0,t) = u(1,t) = 0$ it is clear that we should look for solution in terms of Fourier sine series.

Non homogeneous PDE: Dirichlet boundary condition

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From the boundary conditions $u(0,t) = u(1,t) = 0$ it is clear that we should look for solution in terms of Fourier sine series.

The Fourier sine series of $F(x,t)$ is given by (for $n \geq 1$)

$$\begin{aligned} F_n(t) &= 2\int_0^1 F(x,t)\sin n\pi x\,dx \\ &= 2\int_0^1 e^t \sin n\pi x\,dx \\ &= \frac{2(1-(-1)^n)e^t}{n\pi} \end{aligned}$$

Non homogeneous PDE: Dirichlet boundary condition

Example (continued ...)

Thus, the Fourier series for e^t is given by

$$e^t = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

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Example (continued ...)

Thus, the Fourier series for e^t is given by

$$e^t = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

The Fourier sine series for $f(x) = x(x-1)$ is given by

$$x(x-1) = \sum_{n\geq 1} \frac{4((-1)^n - 1)}{(n\pi)^3} \sin n\pi x$$

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Substitute $u(x,t) = \sum_{n\geq 1} u_n(t) \sin n\pi x$ into the equation $u_t - u_{xx} = e^t$

$$\sum_{n\geq 1} \left(u_n'(t) + n^2\pi^2 u_n(t)\right) \sin n\pi x = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

Non homogeneous PDE: Dirichlet boundary condition

Example (continued ...)

Thus, for $n \geq 1$ and even we get

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

that is,

$$u_n(t) = C_n e^{-n^2\pi^2 t}$$

Non homogeneous PDE: Dirichlet boundary condition

Example (continued ...)

Thus, for $n \geq 1$ and even we get

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

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If $n \geq 1$ and even, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

For $n \geq 1$ odd we get

$$u_n'(t) + n^2\pi^2 u_n(t) = \frac{4}{n\pi} e^t$$

that is,

$$u_n(t) = e^{-n^2\pi^2 t} \int_0^t \frac{4}{n\pi} e^s e^{n^2\pi^2 s} ds + C_n e^{-n^2\pi^2 t}$$

Non homogeneous PDE: Dirichlet boundary condition

Example (continued ...)

If $n \geq 1$ and odd, we have the Fourier coefficient of $x(x-1)$ is $\frac{-8}{(n\pi)^3}$. Thus, we get

$$u_n(0) = C_n = \frac{-8}{(n\pi)^3}$$

Thus, the solution we are looking for is

$$u(x,t) = \sum_{n\geq 0} \left(e^{-(2n+1)^2\pi^2 t} \int_0^t \frac{4}{(2n+1)\pi} e^s e^{(2n+1)^2\pi^2 s} ds + \frac{-8}{((2n+1)\pi)^3} e^{-n^2\pi^2 t} \right) \sin(2n+1)\pi x$$

Non homogeneous PDE: Neumann boundary condition

Let us now consider the following PDE

$$\begin{aligned} &u_t - k^2 u_{xx} = F(x,t) && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = f_1(t) && t > 0 \\ &u_x(L,t) = f_2(t) && t > 0 \\ &u(x,0) = f(x) && 0 \leq x \leq L \end{aligned}$$

How do we solve this?

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How do we solve this?Let us first make the substitution

$$z(x,t) = u(x,t) - (x - \frac{x^2}{2L})f_1(t) - \frac{x^2}{2L}f_2(t)$$

Then clearly

- $z_t - k^2 z_{xx} = G(x,t)$
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Clearly, there is a unique solution to the differential equation $(*)$ with initial condition $(!)$.

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The solution to the above equation is given by

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We can find the constant using the initial condition.

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From the boundary conditions $u_x(0,t) = u_x(1,t) = 0$ it is clear that we should look for solution in terms of Fourier cosine series.

Non homogeneous PDE: Neumann boundary condition

Example

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From the boundary conditions $u_x(0,t) = u_x(1,t) = 0$ it is clear that we should look for solution in terms of Fourier cosine series.

The Fourier cosine series of $F(x,t)$ is given by (for $n \geq 0$)

$$F_0(t) = \int_0^1 F(x,t)\,dx = \int_0^1 e^t dx = e^t$$

$$F_n(t) = 2\int_0^1 F(x,t)\cos n\pi x\,dx = 2\int_0^1 e^t \cos n\pi x\,dx = 0 \quad n > 0$$

Non homogeneous PDE: Neumann boundary condition

Example (continued ...)

Thus, the Fourier series for e^t is simply e^t.

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The Fourier cosine series for $f(x) = x(x-1)$ is given by

$$x(x-1) = -\frac{1}{6} + \sum_{n \geq 1} \frac{2((-1)^n + 1)}{(n\pi)^2} \cos n\pi x$$

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Substitute $u(x,t) = \sum_{n\geq 0} u_n(t) \cos n\pi x$ into the equation $u_t - u_{xx} = e^t$

$$\sum_{n\geq 0} \left(u_n'(t) + n^2\pi^2 u_n(t)\right) \cos n\pi x = e^t$$

Non homogeneous PDE: Neumann boundary condition

Example (continued ...)

Thus, for $n = 0$ we get

$$u_0'(t) = e^t$$

that is,

$$u_0(t) = e^t + C_0$$

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In the case $n = 0$, we have that the Fourier coefficient of $x(x-1)$ is $\frac{-1}{6}$. Thus, when we put $u_0(0) = -\frac{1}{6}$ we get $C = -\frac{7}{6}$.
For $n \geq 1$

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

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Let us now use the initial condition to determine the constants.

Non homogeneous PDE: Neumann boundary condition

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Non homogeneous PDE: Neumann boundary condition

Example (continued ...)

In the case $n \geq 1$ and odd, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

Non homogeneous PDE: Neumann boundary condition

Example (continued ...)

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In the case $n \geq 1$ even, we have the Fourier coefficient of $x(x-1)$ is $\frac{4}{(n\pi)^2}$. Thus, we get

$$C_n = \frac{4}{(n\pi)^2}$$

Non homogeneous PDE: Neumann boundary condition

Example (continued ...)

In the case $n \geq 1$ and odd, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

In the case $n \geq 1$ even, we have the Fourier coefficient of $x(x-1)$ is $\frac{4}{(n\pi)^2}$. Thus, we get

$$C_n = \frac{4}{(n\pi)^2}$$

Thus, the solution we are looking for is

$$u(x,t) = e^t - \frac{7}{6} + \sum_{n\geq 1} \left(\frac{1}{(n\pi)^2} e^{-4n^2\pi^2 t} \right) \cos(2n\pi x)$$