MA-207 Differential Equations II

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Two dimensional Laplace equation

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For example, in today's lecture we will work out the case where

$$\begin{array}{lll} u(x,0) = f(x) & u(x,b) = 0 & 0 \leq x \leq a \\ u(0,y) = 0 & u(a,y) = 0 & 0 \leq y \leq b \end{array}$$

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Let $u(x,y) = X(x)Y(y)$. Then the differential equation becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dirichlet boundary conditions: Finding some solutions

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Since $u(0, y) = X(0)Y(y) = 0$, $u(a, y) = X(a)Y(y) = 0$ and we do not want Y to be identically zero, we get that $X(0) = 0$ and $X(a) = 0$.

This boundary condition on X forces that the constant above should be positive. Let us denote this positive constant by λ^2.

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For every $n \geq 1$, let

$$\lambda_n = \frac{n\pi}{a}$$

Dirichlet boundary conditions: Finding some solutions

For each $n \geq 1$, we have a solution to

$$X''(x) + \lambda_n^2 X(x) = 0$$
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given by

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

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$$Y''(y) - \lambda_n^2 Y(y) = 0$$
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The solutions to the above equation are given by

$$Y_n(y) = \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right).$$

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then we will have solved our Laplace equation with the given boundary conditions.

Dirichlet boundary conditions: Formal solutions

Definition

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\begin{aligned}
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where

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Dirichlet boundary conditions: Actual solution

Theorem

Let f be continuous and piecewise smooth on $[0, a]$ such that $f(0) = f(a) = 0$. Consider the Laplace equation with the boundary conditions

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$$\alpha_5 = 1$$

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Thus, the solution to the above problem is given by

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Laplace equation in polar coordinates

Consider the Dirichlet problem in a disc of radius r

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$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

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Laplace equation in polar coordinates

Example. Solve for harmonic function $u(r, \theta)$ in unit disc i.e.

$$\Delta u(r, \theta) = 0, \quad r < 1, \ \theta \in [0, 2\pi]$$

with boundary condition

$$u(1, \theta) = f(\theta) = \begin{cases} \sin\theta, & \theta \in [0, \pi] \\ 0, & \theta \in [\pi, 2\pi] \end{cases}$$

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Assume $u(r, \theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} R(r)\Theta''(\theta) = 0$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

$$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}, \quad \boxed{r^2R''(r) + rR'(r) - \lambda R(r) = 0}$$

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$$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}, \quad \boxed{r^2R''(r) + rR'(r) - \lambda R(r) = 0}$$

Since $u(r, \theta + 2\pi) = u(r, \theta)$, the functions Θ and Θ' need to be 2π periodic.

Thus for the ODE for Θ, we need to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

The eigenvalues and eigenfunctions for periodic eigenvalue problem in Θ are

$$\lambda_0 = 0, \quad \Theta_0 = 1$$

and for $n \geq 1$,

$$\lambda_n = n^2, \quad \Theta_{n,1}(\theta) = \cos(n\theta), \quad \Theta_{n,2}(\theta) = \sin(n\theta)$$

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Thus, the series solution has the form

$$u(r,\theta) = A_0 + \sum_{n \geq 1} \left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) \right)$$

The boundary condition is

$$u(1,\theta) = f(\theta) = A_0 + \sum_{n\geq 1}(A_n \cos(n\theta) + B_n \sin(n\theta))$$

Hence, A_i and B_i are Fourier coefficients of $f(\theta)$.

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Therefore, the solution is

$$u(r,\theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{1}{4n^2-1}r^{2n}\cos(2n\theta) + \frac{1}{2}r\sin\theta$$

Example. Solve for harmonic function $u(r, \theta)$ in an annulus

$$\Delta u(r,\theta) = 0, \qquad 1 < r < 2,\ \theta \in [0, 2\pi]$$
$$u(1,\theta) = \cos\theta, \quad 0 \le \theta \le 2\pi$$
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This BVP can be interpreted as that for the steady state temperature distribution in an annular region where on the outer boundary the heat flux is prescribed and on the inner boundary, the temperature is prescribed.

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$$u(r, 0) = u(r, 2\pi), \quad u_r(r, 0) = u_r(r, 2\pi)$$

Assume $u(r, \theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

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Since $u(r, \theta + 2\pi) = u(r, \theta)$, the functions Θ and Θ' need to be 2π periodic.

Thus for the ODE for Θ, we need to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

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The eigenvalues and eigenfunctions for periodic eigenvalue problem in Θ are

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Hence the general solution is

$$u(r,\theta) = (A_0 + B_0 \ln r) + \sum_{n\geq 1} (A_n r^n \cos(n\theta) + B_n n r^{-n} \cos(n\theta)) + \sum_{n\geq 1} (C_n r^n \sin(n\theta) + D_n r^{-n} \sin(n\theta))$$

Since

$$u(1,\theta) = \cos\theta, \quad u_r(2,\theta) = \sin 2\theta$$

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$$A_1 + B_1 = 1,\ A_n + B_n = 0\ (n \geq 2),\ C_n + D_n = 0\ (n \geq 1)$$

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$$u_r(r,\theta) = \frac{B_0}{r} + \sum_{n\geq 1} n(A_n r^{n-1} - B_n r^{-n-1})\cos n\theta$$

$$+n(C_n r^{n-1} - D_n r^{-n-1})\sin n\theta$$

Compare with $u_r(2,\theta) = \sin 2\theta$, we get $B_0 = 0,$
$2(2C_2 - 2^{-3}D_2) = 1$

$$A_n 2^{n-1} - B_n 2^{-n-1} = 0\ (n \geq 1), \quad C_n 2^{n-1} - D_n 2^{-n-1} = 0\ (n \neq 2)$$

$$A_0 = 0 = B_0$$

For $n = 1$

$$A_1 + B_1 = 1,\ A_1 - B_1 2^{-2} = 0 \implies A_1 = \frac{1}{5}, B_1 = \frac{4}{5}$$

$$C_1 + D_1 = 0, C_1 - D_1 2^{-2} = 0 \implies C_1 = 0, D_1 = 0$$

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For $n = 2$,

$$A_2 + B_2 = 0,\ A_2 2 - B_2 2^{-3} = 0 \implies A_2 = 0 = B_2$$

$$C_2 + D_2 = 0,\ 2C_2 - \frac{1}{2^3} D_2 = \frac{1}{2} \implies C_2 = \frac{4}{17}, D_2 = \frac{-4}{17}$$

For $n > 2$,

$$A_n + B_n = 0,\ A_n 2^{n-1} - B_n 2^{-n-1} = 0 \implies A_n^1 = 0 = B_n^1$$

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Thus the solution is

$$u(r,\theta) = (\frac{1}{5}r + \frac{4}{5}r^{-1})\cos\theta + (\frac{4}{17}r^2 + \frac{-4}{17}r^{-2})\sin 2\theta$$