MA 2017, Tutorial Sheet-1
Power series and Series solution

1. Find the radius of convergence of the following power series.

   (i) $\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)\,(x-1)^n$, (ii) $\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\,(x+1)^n$,

   (iii) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+k+1)}\,x^n$, (iv) $\sum_{n=1}^{\infty} \frac{n^n}{n!}\,x^n$,

   (v) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^n}\,x^n$, (vi) $\sum_{n=1}^{\infty} \frac{(3n)!}{2^n(n!)^3}\,x^n$,

   (vii) $\sum_{n=0}^{\infty} \frac{n(n+1)}{16^n}\,(x-2)^n$, (viii) $\sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2}\,(x+7)^n$.

2. Let R be the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ and k be a positive integer.

   (i) Show that $\sum_{n=1}^{\infty} a_n x^{2n}$ and $\sum_{n=1}^{\infty} a_n^2 x^n$ have radius of convergence $\sqrt{R}$ and R^2 resp.

   (ii) Show that $\sum_{n=1}^{\infty} a_n x^{kn}$ and $\sum_{n=1}^{\infty} a_n^k x^n$ have radius of convergence $\sqrt[k]{R}$ and R^k resp.

3. Find the radius of convergence R and the interval of convergence (if $R > 0$) of the following power series.

   (i) $\sum_{0}^{\infty} (-1)^n (3n+1)\,(x-1)^{2n+1}$, (ii) $\sum_{0}^{\infty} \frac{n!}{(2n)!}\,(x-1)^{2n}$

   (iii) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(27)^n}\,(x-3)^{3n+2}$, (iv) $\sum_{n=0}^{\infty} \frac{9^n(n+1)}{n+2}\,(x-2)^{2n+2}$

4. Determine the radius of convergence of $\sum_{0}^{\infty} n!x^{n^2}$ and $\sum_{0}^{\infty} x^{n!}$.

5. Suppose $y(x) = \sum_{n=0}^{\infty} a_n\, x^n$. Express the following equations in a power series in x.

   (i) $(2+x)y'' + xy' + 3y$, (ii) $(1+3x^2)y'' + 3x^2y' - 2y$,

   (iii) $(1+2x^2)y'' + (2-3x)y' + 4y$.

6. Let $f(x)$ be the function on $\mathbb{R}$ defined by $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$. Show that f is infinitely differentiable. Show that $f^{(n)}(0) = 0$ for all $n > 0$ and conclude that f is not analytic at 0.

7. Suppose $y(x) = \sum_{n=0}^{\infty} a_n (x+1)^n$ on an open interval around $x = -1$. Find a power series in $(x+1)$ for the equation $xy'' + (4+2x)y' + (2+x)y$.

8. Show that the series $y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2^n n!)^2} x^{2n}$ is a solution of $xy'' + y' + xy = 0$.

9. Find the power series in x for the general solution.

   (i) $(1+x^2)y'' + 6xy' + 6y = 0$; (ii) $(1-x^2)y'' - 8xy' - 12y = 0$;

   (iii) $(1+2x^2)y'' + 7xy' + 2y = 0$.

10. Find the power series in $x-1$ for the general solution of ODE
    $(2+4x-2x^2)y'' - 12(x-1)y' - 12y = 0$.

11. Compute $a_0, a_1, \ldots, a_6$ in the series solution $y = \sum_0^{\infty} a_n x^n$ of the IVP

    (i) $(1+2x^2)y'' + 10xy' + 8y = 0, \quad y(0) = 2, \quad y'(0) = -3$.

    (ii) $(1+2x^2)y'' + xy' + y = 0, \quad y(0) = 2, \quad y'(0) = -1$.

12. Find the power series solution in $x - x_0$ of ODE's

    (i) $y'' - y = 0; \quad x_0 = 3$, (ii) $(1-4x+2x^2)y'' + 10(x-1)y' + 6y = 0; \quad x_0 = 1$.

    (iii) $y'' - (x-3)y' - y = 0, \quad x_0 = 3$.

13. Find the power series solution in x of ODE's

    (i) $(1-2x^3)y'' - 10x^2y' - 8xy = 0$. (ii) (Airy equation) $y'' - xy = 0$,

    (iii) (Tchebychev eqn) $(1-x^2)y'' - xy' + p^2y = 0$. (v) (Hermite eqn) $y'' - x^2y = 0$.

14. Find the coefficients $a_0, \ldots, a_5$ in the series solution in $y = \sum_0^{\infty} a_n(x+1)^n$ of the IVP
    $(3+x)y'' + (1+2x)y' - (2-x)y = 0; \quad y(-1) = 2, \; y'(-1) = -3$.

15. Find the coefficients $a_0, \ldots, a_5$ in the series solution in $y = \sum_0^{\infty} a_n x^n$ of the IVP
    $y'' + 3xy' + (4+2x^2)y = 0; \quad y(0) = 2, \; y'(0) = -3$.