MA 2017, Tutorial Sheet-2
Legendre equation and Legendre polynomials

1. Express x^2, x^3, and x^4 as a linear combination of the Legendre polynomials.

2. Prove the following. First equality is Rodrigues formula.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \sum_{m=0}^{[n/2]}(-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}\, x^{n-2m}$$

where $[n/2]$ denotes the greatest integer less than or equal to $n/2$.

3. Show that if $f(x)$ is a polynomial with double (or multiplicity 2) roots at a and b, then $f''(x)$ vanishes at least twice in (a,b). (This is also true if $f(x)$ is a smooth function.)

   Generalize this and show (using Rodrigues' formula) that $P_n(x)$ has n distinct roots in $(-1,1)$.

4. Take the Rodrigues formula as the definition for $P_n(x)$, and show the following relations.

   (i) $P_n(-x) = (-1)^n P_n(x)$, (ii) $P_n'(-x) = (-1)^{n+1}P_n'(x)$,

   (iii) $P_n(1) = 1$, (iv) $P_n(-1) = (-1)^n$,

   (v) $P_{2n+1}(0) = 0$, (vi) $P_{2n}(0) = (-1)^n \dfrac{(2n)!}{2^{2n}(n!)^2}$,

   (vii) $P_n'(1) = \dfrac{1}{2}n(n+1)$, (viii) $P_n'(-1) = (-1)^{n-1}\frac{1}{2}n(n+1)$.

   (ix) $P_{2n}'(0) = 0$, (x) $P_{2n+1}'(0) = (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2}$.

5. Show that $\displaystyle\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx = \begin{cases} \dfrac{2n(n+1)}{2n+1} & \text{if } m=n, \\ 0 & \text{otherwise.} \end{cases}$

6. Show the following relations when $n-m$ is even and nonnegative.

   (a) $\displaystyle\int_{-1}^{1} P_m' P_n' dx = m(m+1)$, (b) $\displaystyle\int_{-1}^{1} x^m P_n'(x) dx = 0$.

   What is the value of the integral if $n-m$ is odd (instead of even)?

7. If $x^n = \displaystyle\sum_{r=0}^{n} a_r P_r(x)$, then show that $a_n = \dfrac{2^n (n!)^2}{(2n)!}$.

8. Expand the following functions $f(x)$ in a series of Legendre polynomials:

$$f(x) \approx \sum_{n \geq 0} c_n P_n \quad \text{with} \quad c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx.$$

The Rodrigues formula is useful to evaluate these integrals. The Legendre expansion theorem (stated in the lecture notes) applies in each case.

(a) $f_1(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$ (b) $f_2(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$

(c) $f_3(x) = \begin{cases} -x & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1. \end{cases}$ (d) $f_4(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases}$

9. If $p(x)$ is a polynomial of degree $n \geq 1$ such that $\int_{-1}^{1} x^k p(x)\, dx = 0$ for $k = 0, 1, \ldots, n-1$, show that $p(x) = cP_n(x)$ for some constant c.