MA 2017, Tutorial Sheet-4
Bessel equation and Bessel functions

1. Using the indicated substitutions, reduce the following differential equations to the Bessel equation and find the general solution in term of the Bessel functions.

   (a) $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0, \quad (\lambda x = z)$.

   (b) $xy'' - 5y' + xy = 0, \quad (y = x^3u)$.

   (c) $y'' + k^2xy = 0, \quad (y = u\sqrt{x},\ \frac{2}{3}kx^{3/2} = z)$.

   (d) $x^2y'' + (1-2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0, \quad (y = x^\nu u,\ \ x^\nu = z)$.

2. Prove the following Bessel identities.

   (a) $[x^pJ_p(x)]' = x^pJ_{p-1}(x)$

   (b) $[x^{-p}J_p(x)]' = -x^{-p}J_{p+1}(x)$.

   (c) $J_p'(x) + \dfrac{p}{x}J_p(x) = J_{p-1}(x)$.

   (d) $J_p'(x) - \dfrac{p}{x}J_p(x) = -J_{p+1}(x)$.

   (e) $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$.

   (f) $J_{p-1}(x) + J_{p+1}(x) = \dfrac{2p}{x}J_p(x)$.

3. Show that (a) $J_{1/2} = \sqrt{\dfrac{2}{\pi x}}\sin x$, (b) $J_{-1/2} = \sqrt{\dfrac{2}{\pi x}}\cos x$.

   (c) $J_{3/2} = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\sin x}{x} - \cos x\right)$, (d) $J_{-3/2} = -\sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\cos x}{x} + \sin x\right)$.

4. For an integer n, show that $J_n(x)$ is an even (resp. odd) function if n is even (resp. odd).

5. Express $J_2(x), J_3(x), J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

6. Show the following.

   (a) $J_3 + 3J_0' + 4J_0''' = 0$. (b) $\displaystyle\int J_{\nu+1}dx = \int J_{\nu-1}dx - 2J_\nu$.

7. If y_1 and y_2 are any two solutions of the Bessel equation of order ν, then show that $y_1y_2' - y_1'y_2 = c/x$ for a suitable constant c.

8. Show that $\displaystyle\int x^\mu J_\nu(x)\,dx = x^\mu J_{\nu+1}(x) - (\mu - \nu - 1)\int x^{\mu-1}J_{\nu+1}(x)\,dx$.

9. Expand the indicated function in Fourier-Bessel series over the given interval and in terms of the Bessel function of given order. (The Bessel expansion theorem applies in each case.)

   (a) $f(x) = 1$ over $[0, 1]$, $\nu = 0$.

   (b) $f(x) = x$ over $[0, 1]$, $\nu = 1$.

   (c) $f(x) = x^3$ over $[0, 1]$, $\nu = 1$.

   (d) $f(x) = x^2$ over $[0, 1]$, $\nu = 2$.

   (e) $f(x) = \sqrt{x}$ over $[0, 1]$, $\nu = \frac{1}{2}$.

10. If $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ 1/2 & \text{if } x = 1/2 \\ 0 & \text{if } 1/2 < x \le 1 \end{cases}$, show that $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{J_1(\lambda_{0,n}/2)}{\lambda_{0,n} J_1(\lambda_{0,n})^2} J_0(\lambda_{0,n} x)$,
    where $\lambda_{0,n}$'s are positive zeros of $J_0(x)$.