De Rham Cohomology of some spaces

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Throughout the document $H^*(X)$ is used in place of $H^*(X, \mathbb{R})$ to denote the de Rham cohomology of the smooth manifold X.

1 De Rham cohomology of $S^2 \setminus \{x_1, x_2, \ldots, x_k\}$

Let $X = \mathbb{R}^2 \setminus \{x_1, x_2, \ldots, x_{k-1}\}$. Then S^2 minus k points is diffeomorphic to X, thus it suffices to compute the cohomology of X. We will compute the cohomology groups of X using the Mayer-Vietoris sequence. Let V be the disjoint union of $k-1$ discs around the $k-1$ points x_i in $\mathbb{R}^2$. Then $\mathbb{R}^2 = X \cup V$, and $X \cap V$ is the disjoint union of $k-1$ punctured disks around the points x_i. Using Mayer-Vietoris Sequence for $\mathbb{R}^2 = X \cup V$, we have the long exact sequence,

$$\begin{aligned} 0 \to & H^0(\mathbb{R}^2) \to H^0(X) \oplus H^0(V) \to H^0(X \cap V) \to \\ & H^1(\mathbb{R}^2) \to H^1(X) \oplus H^1(V) \to H^1(X \cap V) \to \\ & H^2(\mathbb{R}^2) \to H^2(X) \oplus H^2(V) \to H^2(X \cap V) \to 0 \end{aligned}$$

Now

$$H^i(V) = \begin{cases} \mathbb{R}^{k-1} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^i(X \cap V) = \begin{cases} \mathbb{R}^{k-1} & \text{if } i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Using the above and replacing particular values in the long exact sequence we get the long exact sequence

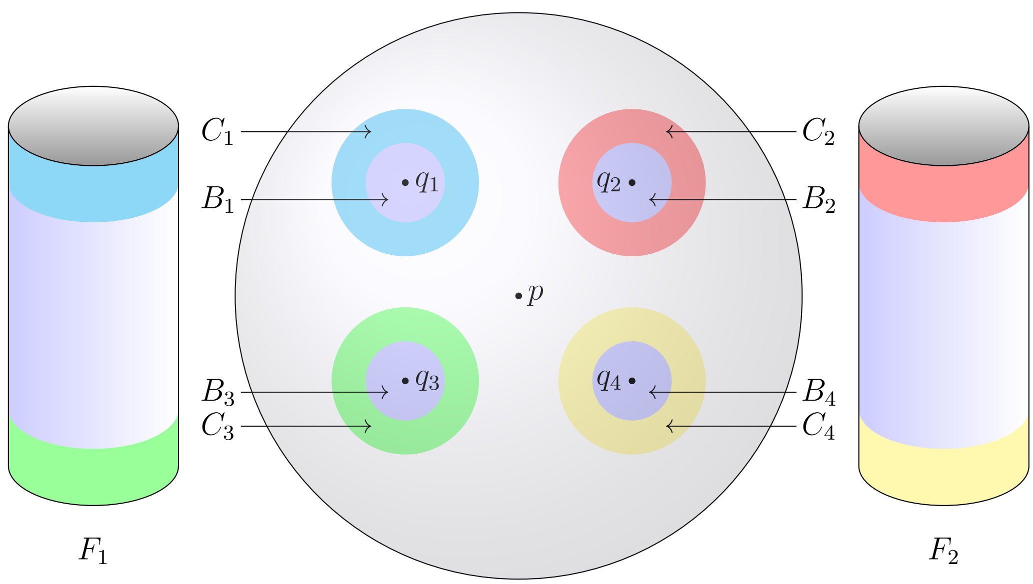
$$\begin{aligned}0 \to &\mathbb{R} \to \mathbb{R} \oplus \mathbb{R}^{k-1} \to \mathbb{R}^{k-1} \to \\ &0 \to H^1(X) \oplus 0 \to \mathbb{R}^{k-1} \to \\ &0 \to H^2(X) \oplus 0 \to 0 \to 0\end{aligned}$$

So we get,

$$H^i(X) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ \mathbb{R}^{k-1} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.0.1}$$

2 De Rham cohomology of $\Sigma_g \setminus \{p\}$

Let Σ_g denote the compact orientable smooth genus g surface. Let $M := \Sigma_g \setminus \{p\}$, where p is a point on Σ_g. We know M can be constructed by attaching g-handles to $S^2 \setminus \{p\}$. Let $\{q_i\}$ for $1 \leq i \leq 2g$ be $2g$ many points on S^2 different from p. Around each q_i we take two small open discs B_i and C_i such that $B_i \subset C_i \subset S^2 \setminus \{p\}$ and $C_i \cap C_j = \phi$ for $i \neq j$. Let $U = (S^2 \setminus \{p\}) \setminus \bigsqcup_{i=1}^{2g} B_i$. Let $F_1, F_2, \ldots, F_g$ denote g-handles, that is, $F_i \cong S^1 \times (0,1)$ and we attach each F_i to U such that $U \cap F_i = (C_i \setminus B_i) \bigsqcup (C_{i+g} \setminus B_{i+g})$ for $i = 1, 2, \ldots g$. Let $V = \bigsqcup_{i=1}^{g} F_i$.



The above picture illustrates the situation when $g = 2$. Regions of the same color are being glued in a manner so that we get the surface which is a sphere with 2 handles.

So we have U and V are open subsets of M such that, $M = U \cup V$ and $U \cap V = \bigsqcup_{i=1}^{2g}(C_i \setminus B_i)$. Let $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$ be the inclusion maps. Using Mayer-Vietoris Sequence we have the exact sequence,

$$
\begin{aligned}
0 \to & H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to \\
& H^1(M) \to H^1(U) \oplus H^1(V) \xrightarrow{j_U^* - j_V^*} H^1(U \cap V) \to \\
& H^2(M) \to H^2(U) \oplus H^2(V) \xrightarrow{j_U^* - j_V^*} H^2(U \cap V) \to 0
\end{aligned} \tag{2.0.1}
$$

We have $H^0(M) = \mathbb{R}$ since M is connected. The smooth manifold U is diffeomorphic to $\mathbb{R}^k$ minus $2g$ points. Thus, using equation (1.0.1) we get

$$
H^k(U) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \mathbb{R}^{2g} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}
$$

$$
H^k(V) = \begin{cases} \mathbb{R}^{g} & \text{if } k = 0 \\ \mathbb{R}^{g} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}
$$

$$
H^k(U \cap V) = \begin{cases} \mathbb{R}^{2g} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}
$$

Now consider the map $j_U^* - j_V^* : H^1(U) \oplus H^1(V) \to H^1(U \cap V)$.

Lemma 2.0.2. *$j_U^* - j_V^* : H^1(U) \oplus H^1(V) \to H^1(U \cap V)$ is surjective.*

Proof. Let $W = \bigsqcup_{i=1}^{2g} C_i$. Then we have U and W are open is $S^2 \setminus \{p\}$ such that $U \cup W = S^2 \setminus \{p\}$ and $U \cap W$ is diffeomorphic to $\bigsqcup_{i=0}^{2g}(C_i \setminus B_i) = U \cap V$. If $f_U : U \cap W \to U$ and $f_W : U \cap W \to W$ denote the inclusion maps, then $j_U = f_U$. Using mayer-Vietoris sequence on $S^2 = U \cup W$, we have the long exact sequence,

$$
\begin{aligned}
0 \to & H^0(S^2 \setminus \{p\}) \to H^0(U) \oplus H^0(W) \to H^0(U \cap W) \to \\
& H^1(S^2 \setminus \{p\}) \to H^1(U) \oplus H^1(W) \xrightarrow{f_U^* - f_W^*} H^1(U \cap W) \to \\
& H^2(S^2 \setminus \{p\}) \to H^2(U) \oplus H^2(W) \xrightarrow{f_U^* - f_W^*} H^2(U \cap W) \to 0
\end{aligned}
$$

Now $H^2(S^2 \setminus \{p\}) = 0$ and $H^1(W) = 0$. So we have exact sequence,

$$0 \to \cdots \to H^1(U) \xrightarrow{f_U^*} H^1(U \cap W) \to 0$$

So $f_U^* : H^1(U) \to H^1(U \cap W)$ is surjective and hence $j_U^* : H^1(U) \to H^1(U \cap V)$ is also sujective. So $j_U^* - j_V^* : H^1(U) \oplus H^1(V) \to H^1(U \cap V)$ is surjective. □

Combining the above Lemma with equation (2.0.1) we get the long eaxct sequence,

$$\begin{aligned} 0 \to & \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}^g \to \mathbb{R}^{2g} \to \\ & H^1(M) \to \mathbb{R}^{2g} \oplus \mathbb{R}^g \xrightarrow{j_U^* - j_V^*} \mathbb{R}^{2g} \to \\ & H^2(M) \to 0 \to 0 \to 0 \end{aligned}$$

Since $j_U^* - j_V^*$ is surjective we get that $H^2(M) = 0$ and $H^1(M) = \mathbb{R}^{2g}$. So

$$H^k(\Sigma_g \setminus \{p\}) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \mathbb{R}^{2g} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

3 De Rham cohomology of Σ_g

Let $p \in \Sigma_g$. Take $U = \Sigma_g \setminus \{p\}$ and V be a small disc around p in Σ_g. So $U \cup V = \Sigma_g$ and $U \cap V = V \setminus \{p\}$. Using Mayer Vietoris sequence and cohomology of U and V, we get the long exact sequence,

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to H^1(\Sigma_g) \to \mathbb{R}^{2g} \to \mathbb{R} \to H^2(\Sigma_g) \to 0\,. \tag{3.0.1}$$

By the surjection on the right we conclude that $H^2(\Sigma_g)$ is either $\mathbb{R}$ or 0.

Lemma 3.0.2. $H^2(\Sigma_g) = \mathbb{R}$.

Proof. Let f be a compactly supported non-negative non-zero funtion on $V (\cong \mathbb{R}^2)$. Let $\omega = f dx \wedge dy$ be a 2-form on V. Since ω is compactly supported, ω can be taken as a 2-form on Σ_g by extending as 0 outside V. So $[\omega] \in H^2(\Sigma_g)$ since it is a top form. Consider the linear map $\int : H^2(\Sigma_g) \to \mathbb{R}$. Then $\int [\omega] \neq 0$ and hence $H^2(\Sigma_g)$ is nontrivial. Again we already observed that $H^2(\Sigma_g)$ is either $\mathbb{R}$ or 0, so we conclude that $H^2(\Sigma_g) = \mathbb{R}$. □

So from the long exact sequence (3.0.1) we get $H^2(\Sigma_g) = \mathbb{R}$ and $H^1(\Sigma_g) = \mathbb{R}^{2g}$. So,

$$H^k(\Sigma_g) = \begin{cases} \mathbb{R} & \text{if } k = 0, 2 \\ \mathbb{R}^{2g} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

4 De Rham cohomology of $\mathbb{CP}^n$

We know $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$. So,

$$H^k(\mathbb{CP}^1) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}$$

Consider $\mathbb{CP}^n$ for $n > 1$. Since $\mathbb{CP}^n$ is connected, $H^0(\mathbb{CP}^n) \cong \mathbb{R}$. Now $\mathbb{CP}^{n-1}$ can be identified with the subset

$$\{[z_0 : z_1 : \ldots : z_n] \in \mathbb{CP}^n \mid z_n = 0\}$$

of $\mathbb{CP}^n$. Let

(1) $p := [0 : 0 : \ldots : 0 : 1]$,

(2) $U := \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$,

(3) $V := \mathbb{CP}^n \setminus \{p\}$.

Then $U \cup V = \mathbb{CP}^n$ and $U \cap V = U \setminus \{p\}$. Using the Mayer-Vietoris sequence we have the exact sequence,

$$\begin{aligned} 0 \to & H^0(\mathbb{CP}^n) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to \\ & H^1(\mathbb{CP}^n) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to \\ & \ldots \\ & H^{2n}(\mathbb{CP}^n) \to H^{2n}(U) \oplus H^{2n}(V) \to H^{2n}(U \cap V) \to 0 \end{aligned} \tag{4.0.1}$$

Now we observe that U is diffeomorphic to $\mathbb{C}^n$. This shows that $H^k(U) = 0$ if $k \neq 0$. Similarly, $U \cap V$ is diffeomorphic to $\mathbb{C}^n \setminus \{0\}$ which deformation retracts onto S^{2n-1} Hence $H^k(U \cap V) = 0$ unless $k \in \{0, 2n-1\}$ and $H^0(U \cap V) = H^{2n-1}(U \cap V) = \mathbb{R}$.

Lemma 4.0.2. *V deformation retracts to $\mathbb{CP}^{n-1}$ smoothly.*

Proof. We define a homotopy $F : V \times \mathbb{R} \to V$ by

$$([z_0 : \ldots : z_n], t) \mapsto [z_0 : \ldots : z_{n-1} : (1-t)z_n)] \,. \tag{4.0.3}$$

To see that F is smooth : We cover $\mathbb{CP}^n$ by the open sets $U_0, \ldots, U_n$ where $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n | z_i \neq 0\}$. Then each U_i is diffeomorphic to $\mathbb{C}^n$ by the diffeomorphism :

$$\varphi_i : U_i \to \mathbb{C}^n$$

$$[z_0 : \cdots : z_n] \mapsto \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)$$

We observe that $[z_0 : \cdots : z_n] \in V$ iff $(z_0, \ldots, z_{n-1}) \neq (0, \ldots, 0)$. So if $(q,t) \in V \times \mathbb{R}$, without loss of generality we can assume $(q,t) \in (U_j \cap V) \times \mathbb{R}$ for $j \in \{0, \ldots, n-1\}$ and also $F(q,t) \in (U_j \cap V)$. Now $(U_j \cap V) \times \mathbb{R}$ is diffeomorphic to an open subset of $\mathbb{C}^n \times \mathbb{R}$ via the map $(\varphi_j|_{U_j \cap V} \times id)$. So

$$\varphi_j \circ F \circ (\varphi_j^{-1} \times id) : \varphi_j(U_j \cap V) \times \mathbb{R} \to \mathbb{C}^n$$

is a map between euclidean spaces given by,

$$(w_1, \ldots, w_n, t) \mapsto (w_1, \ldots, w_{n-1}, (1-t)w_n) \,.$$

This map is clearly smooth and hence F is also smooth.

Recall that we identified $\mathbb{CP}^{n-1}$ with the subset $\{[z_0 : z_1 : \ldots : z_n] \in \mathbb{CP}^n \mid z_n = 0\}$ of $\mathbb{CP}^n$. So it is also clear that for all $v \in V$ we have $F(v,0) = v$ and $F(v,1) \in \mathbb{CP}^{n-1}$. Further, for all $w \in \mathbb{CP}^{n-1}$ we have $F(w,t) = w$. Hence, V deformation retracts to $\mathbb{CP}^{n-1}$ smoothly. ◻

By induction let us assume that $H^k(\mathbb{CP}^{n-1}) = \mathbb{R}$ when k is an even integer in $[0, 2n-2]$, and 0 for other k. Thus, the same result follows for V, that is,

$$H^k(V) = \begin{cases} \mathbb{R} & \text{if } k \in [0, 2n-2] \text{ and } k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathbb{CP}^n$, U, V and $U \cap V$ are all connected, it follows that the sequence

$$0 \to H^0(\mathbb{CP}^n) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to 0$$

is exact. From (4.0.1) it follows that

$$
\begin{aligned}
(4.0.4) \qquad 0 \to & H^1(\mathbb{CP}^n) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to \\
& \dots \\
& H^{2n-1}(\mathbb{CP}^n) \to H^{2n-1}(U) \oplus H^{2n-1}(V) \to H^{2n-1}(U \cap V) \to \\
& H^{2n}(\mathbb{CP}^n) \to H^{2n}(U) \oplus H^{2n}(V) \to H^{2n}(U \cap V) \to 0
\end{aligned}
$$

is an exact sequence. Since $H^i(U \cap V)$ vanishes for $1 \leq i \leq 2n-2$ we get that $H^i(\mathbb{CP}^n) \to H^i(V)$ is an isomorphism for $1 \leq i \leq 2n-2$. The last part of the above sequence is

$$
\begin{aligned}
0 \to & H^{2n-1}(\mathbb{CP}^n) \to H^{2n-1}(U) \oplus H^{2n-1}(V) \to H^{2n-1}(U \cap V) \to \\
& H^{2n}(\mathbb{CP}^n) \to H^{2n}(U) \oplus H^{2n}(V) \to H^{2n}(U \cap V) \to 0
\end{aligned}
$$

Again, substituting specifc values and using induction hypothesis this becomes

$$0 \to H^{2n-1}(\mathbb{CP}^n) \to 0 \to \mathbb{R} \to H^{2n}(\mathbb{CP}^n) \to 0\,.$$

This shows that $H^{2n-1}(\mathbb{CP}^n) = 0$ and $H^{2n}(\mathbb{CP}^n) = \mathbb{R}$. Thus, by induction we see that

$$H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & \text{if } k \in [0, 2n] \text{ and } k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

5 De Rham cohomology of $\mathbb{RP}^n$

We know $\mathbb{RP}^1$ is diffeomorphic to S^1. So,

$$H^k(\mathbb{RP}^1) = \begin{cases} \mathbb{R} \text{ if } k = 0 \text{ or } 1 \\ 0 \text{ otherwise} \end{cases}$$

Let $n > 1$. Since $\mathbb{RP}^n$ is connected, $H^0(\mathbb{RP}^n) \cong \mathbb{R}$.

Lemma 5.0.1. $H^k(\mathbb{RP}^n) = 0$ *for* $0 < k < n$.

Proof. Let $\pi : S^n \to \mathbb{RP}^n$ be the quotient map. We have the induced map $\pi^* : H^k(\mathbb{RP}^n) \to H^k(S^n)$. Let $0 < k < n$ and $[\sigma] \in H^k(\mathbb{RP}^n)$. Let $\sigma \in \Gamma(\mathbb{RP}^n, \wedge^k \Omega_{\mathbb{RP}^n})$ represent the class $[\sigma]$. The class $\pi^*[\sigma]$ is represented by the form $\pi^*\sigma$. Now $H^k(S^n) = 0$ for $0 < k < n$ and so $[\pi^*\sigma] = 0$. Thus, there is an $\eta \in \Gamma(S^n, \wedge^{k-1}\Omega_{S^n})$ such that $d(\eta) = \pi^*\sigma$. Let $f : S^n \to S^n$ be the involution $x \mapsto -x$. Then we have $\pi \circ f = \pi$. Thus, $f^*\pi^*\sigma = \pi^*\sigma$. Let

$$\omega := \frac{\eta + f^*\eta}{2}.$$

Then

$$\begin{aligned} d\omega &= d\left(\frac{\eta + f^*\eta}{2}\right) \\ &= \frac{1}{2}(d\eta + f^*d\eta) \\ &= \frac{1}{2}(\pi^*\sigma + f^*\pi^*\sigma) \\ &= \frac{1}{2}(\pi^*\sigma + \pi^*\sigma) = \pi^*\sigma. \end{aligned}$$

We notice that $D\pi_x$ is an isomorphism for all $x \in S^n$. We define

$$\tilde{\omega} : S^n \to \wedge^{k-1}\Omega_{\mathbb{RP}^n}$$

by

$$\tilde{\omega}(x) = (\wedge^{k-1}((D\pi_x)^{\check{}})^{-1})(\omega(x)).$$

Now we observe that $\tilde{\omega} \circ f = \tilde{\omega}$. That is, the section $\tilde{\omega}$ is invariant under the involution f and so descends to a section of $\mathbb{RP}^n$. So we get an induced map,

$$\tilde{\omega}_0 : \mathbb{RP}^n \to \wedge^{k-1}\Omega_{\mathbb{RP}^n}$$

and clearly $\tilde{\omega}_0 \in \Gamma(\mathbb{RP}^n, \wedge^{k-1}\Omega_{\mathbb{RP}^n})$ and is such that $\pi^*\tilde{\omega}_0 = \tilde{\omega}$. Next we want to check that $d\tilde{\omega}_0 = \sigma$. But this is a local check and since π is a local diffeomorphism, it suffices to check that $\pi^*d\tilde{\omega}_0 = \pi^*\sigma$. But this is obvious. This shows that $[\sigma] = 0$ in $H^k(\mathbb{RP}^n)$. Hence $H^k(\mathbb{RP}^n) = 0$ for $0 < k < n$. □

In view of the above Lemma it only remains to compute $H^n(\mathbb{RP}^n)$. We will do this using the Mayer-Vietoris sequence. Let

(1) $p := [0 : 0 : \ldots : 0 : 1]$,

(2) $U := \mathbb{RP}^n \setminus \mathbb{RP}^{n-1}$,

(3) $V := \mathbb{RP}^n \setminus \{p\}$.

Then $U \cup V = \mathbb{RP}^n$ and $U \cap V = U \setminus \{p\}$. Clearly U is diffeomorphic to $\mathbb{R}^n$, V deformation retracts to $\mathbb{RP}^{n-1}$ (same homotopy as in equation (4.0.3)) and $U \cap V$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$ which deformation retracts onto S^{n-1}. So $H^k(U) \cong 0$, $H^k(V) \cong H^k(\mathbb{RP}^{n-1})$ and $H^k(U \cap V) \cong H^k(S^{n-1})$ for all $k > 0$. Using Mayer-Vietoris sequence we get the following exact sequnce

$$0 \to \cdots \to H^{n-1}(\mathbb{RP}^n) \to H^{n-1}(\mathbb{RP}^{n-1}) \to H^{n-1}(S^{n-1}) \to H^n(\mathbb{RP}^n) \to 0$$

Using $H^{n-1}(\mathbb{RP}^n) \cong 0$, we have the s.e.s,

$$0 \to H^{n-1}(\mathbb{RP}^{n-1}) \to \mathbb{R} \to H^n(\mathbb{RP}^n) \to 0$$

Using $H^1(\mathbb{RP}^1) \cong \mathbb{R}$, by induction we get

$$H^n(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

6 De Rham cohomology of $\Sigma_g \setminus \{x_1, x_2, \ldots, x_k\}$

Let X denote the manifold $\Sigma_g \setminus \{x_1, x_2, \ldots, x_k\}$. Let $M = \Sigma_g \setminus \{x_1\}$ and V be disjoint union of $k-1$ small discs around $x_2, \ldots, x_k$. So $X \cup V = M$ and $X \cap V$ is disjoint union of $k-1$ annuli. We have,

$$H^i(M) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ \mathbb{R}^{2g} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H^i(V) = \begin{cases} \mathbb{R}^{k-1} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^i(X \cap V) = \begin{cases} \mathbb{R}^{k-1} & \text{if } i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Using M-V sequence we have the long exact sequence:

$$\begin{aligned} 0 \to & H^0(M) \to H^0(X) \oplus H^0(V) \to H^0(X \cap V) \to \\ & H^1(M) \to H^1(X) \oplus H^1(V) \to H^1(X \cap V) \to \\ & H^2(M) \to H^2(X) \oplus H^2(V) \to H^2(X \cap V) \to 0 \end{aligned}$$

Plugging in the values we get

$$\begin{aligned}0 \to &\mathbb{R} \to \mathbb{R} \oplus \mathbb{R}^{k-1} \to \mathbb{R}^{k-1} \to \\ &\mathbb{R}^{2g} \to H^1(X) \to \mathbb{R}^{k-1} \to \\ &0 \to H^2(X) \to 0 \to 0\end{aligned}$$

So we clearly get,

$$H^i(X) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ \mathbb{R}^{2g+k-1} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

7 De Rham cohomology of $\Sigma_g \# \mathbb{RP}^2$

Let $M = \Sigma_g \# \mathbb{RP}^2$. Let $p \in \Sigma_g$ and $q \in \mathbb{RP}^2$. Let $C \subset \Sigma_g$ be a disc around p and $D \subset \mathbb{RP}^2$ be a disc around q. Let $U = \Sigma_g \setminus C$ and $V = \mathbb{RP}^2 \setminus D$. Then we can get M by gluing U and V so that $U \cap V$ is diffeomorphic to an annulus. Let $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$ denote the inclusion maps. Using Mayer-Vietoris Sequence we have the exact sequence,

(7.0.1)
$$\begin{aligned}0 \to &H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to \\ &H^1(M) \to H^1(U) \oplus H^1(V) \xrightarrow{j_U^* - j_V^*} H^1(U \cap V) \to \\ &H^2(M) \to H^2(U) \oplus H^2(V) \to H^2(U \cap V) \to 0\end{aligned}$$

We have $H^0(M) = \mathbb{R}$ since M is connected. Now U is diffeomorphic to $\Sigma_g \setminus \{p\}$, so

$$H^k(U) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \mathbb{R}^{2g} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

V is diffeomorphic to $\mathbb{RP}^2 \setminus \{q\}$, hence deformation retracts to $\mathbb{RP}^1 \cong S^1$, as we saw earlier .

$$H^k(V) = \begin{cases} \mathbb{R} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and clearly

$$H^k(U \cap V) = \begin{cases} \mathbb{R} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Now consider the map $j_V^*: H^1(V) \to H^1(U \cap V)$.

Lemma 7.0.2. *$j_V^*: H^1(V) \to H^1(U \cap V)$ is surjective.*

Proof. Let $W \subset \mathbb{RP}^2$ be the set $D \cup (U \cap V)$. Then W is an open disk around q. Then $\mathbb{RP}^2 = V \cup W$ and $V \cap W = U \cap V$ is an annulus around q. Let $f_V : V \cap W \to V$ and $f_W : V \cap W \to W$ denote the inclusions. Then $f_V = j_V$. Using Mayer-Vietoris sequence on $\mathbb{RP}^2 = V \cup W$, we have the long exact sequence,

$$\begin{aligned} 0 \to & H^0(\mathbb{RP}^2) \to H^0(V) \oplus H^0(W) \to H^0(V \cap W) \to \\ & H^1(\mathbb{RP}^2) \to H^1(V) \oplus H^1(W) \xrightarrow{f_V^* - f_W^*} H^1(V \cap W) \to \\ & H^2(\mathbb{RP}^2) \to H^2(V) \oplus H^2(W) \xrightarrow{f_V^* - f_W^*} H^2(V \cap W) \to 0 \end{aligned}$$

Now $H^2(\mathbb{RP}^2) = 0$ and $H^1(W) = 0$. So we have exact sequence,

$$0 \to \cdots \to H^1(V) \xrightarrow{f_V^*} H^1(V \cap W) \to 0$$

So $f_V^* : H^1(V) \to H^1(V \cap W)$ is surjective and hence $j_V^* : H^1(V) \to H^1(U \cap V)$ is also surjective. ☐

So $j_U^* - j_V^* : H^1(U) \oplus H^1(V) \to H^1(U \cap V)$ is surjective. Combining the above lemma with (7.0.1) we get the long eaxct sequence,

$$\begin{aligned} 0 \to & \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to \\ & H^1(M) \to \mathbb{R}^{2g} \oplus \mathbb{R} \xrightarrow{j_U^* - j_V^*} \mathbb{R} \to \\ & H^2(M) \to 0 \to 0 \to 0 \end{aligned}$$

Since $j_U^* - j_V^*$ is surjective we get that $H^2(M) = 0$ and $H^1(M) = \mathbb{R}^{2g}$. So

$$H^k(\Sigma_g \# \mathbb{RP}^2) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \mathbb{R}^{2g} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

8 Some constructions out of vector spaces and their universal properties

8.1 Construction of $\mathbb{P}^n_{\mathbb{C}}$ and the tautological line bundle

We first construct $\mathbb{P}^n_{\mathbb{C}}$ using gluing. Let $U_i \cong \mathbb{C}^n$ be the set

$$U_i = \{z = (z_0 : \ldots : z_{i-1} : 1 : z_{i+1} : \ldots : z_n) \in \mathbb{C}^{n+1}\}\,.$$

Let $U_{ij} \subset U_i$ be the set of those points such that $z_j \neq 0$. In particular, this means that $U_{ii} = U_i$. Define maps $\varphi_{ij} : U_{ij} \to U_{ji}$ by multiplication by $\frac{1}{z_j}$, that is,

$$\varphi_{ij}(z) = (\frac{z_0}{z_j} : \ldots : \frac{z_{i-1}}{z_j} : \frac{1}{z_j} : \frac{z_{i+1}}{z_j} : \ldots : \frac{z_n}{z_j})\,.$$

It is easily checked that these satisfy the cocycle condition, and gluing these we get $\mathbb{P}^n_{\mathbb{C}}$.

Next we will glue to construct a line sub-bundle of $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$. First note that over U_i we have a canonical line sub-bundle, given by the following section of $U_i \times \mathbb{C}^{n+1} \to U_i$,

$$z \mapsto s_i(z) = (z_0 : \ldots : z_{i-1} : 1 : z_{i+1} : \ldots : z_n)\,.$$

It is clear that this section is smooth and non-vanishing, thus, it defines a line sub-bundle of $U_i \times \mathbb{C}^{n+1}$. Let us denote this line sub-bundle by $\mathcal{L}_i$. As above, the map multiplication by $\frac{1}{z_j}$ fits into a commutative diagram of smooth maps

$$\begin{array}{ccc} \mathcal{L}_i|_{U_{ij}} & \longrightarrow & \mathcal{L}_j|_{U_{ji}} \\ \downarrow & & \downarrow \\ U_{ij} & \longrightarrow & U_{ji} \end{array}$$

The top horizontal arrow is given by $(z, v) \mapsto (\varphi_{ij}(z), v/z_j)$. As a result we get that the $\mathcal{L}_i$'s glue together to give a line sub-bundle of $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$. We denote this line sub-bundle by $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(-1)$. It has the description that over a point $[v] \in \mathbb{P}^n_{\mathbb{C}}$ the fiber is exactly the line in $\mathbb{C}^{n+1}$ corresponding to $[v]$.

8.2 Universal property of $\mathbb{P}(V)$

Let V be the vector space $\mathbb{C}^{n+1}$. Recall that on $\mathbb{P}(V)$ we have a short exact sequence of vector bundles, namely,

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathbb{P}(V) \times \mathbb{C}^{n+1} \to \mathcal{Q} \to 0\,.$$

The above line sub-bundle has the following property. Given a point $p \in \mathbb{P}(V)$, it corresponds to a line $[l_p] \in \mathbb{C}^{n+1}$. When we restrict the inclusion $\mathcal{O}_{\mathbb{P}(V)}(-1) \hookrightarrow \mathbb{P}(V) \times \mathbb{C}^{n+1}$ over the point p, we get exactly the line l_p inside $\mathbb{C}^{n+1}$.

Let $f : X \to \mathbb{P}(V)$ be a smooth map. The pull back of the above short exact sequence to X along f gives a short exact sequence of vector bundles on X

$$0 \to f^*(\mathcal{O}_{\mathbb{P}(V)}(-1)) \to X \times \mathbb{C}^{n+1} \to f^*\mathcal{Q} \to 0\,.$$

Thus, consider the map Φ between the following two sets

$$\Phi : \{\text{Smooth maps } X \to \mathbb{P}^n\} \to \{\text{Line sub-bundles of } X \times \mathbb{C}^{n+1}\}\,,$$

defined as $\Phi(f)$ is the line sub-bundle $f^*(\mathcal{O}_{\mathbb{P}(V)}(-1)) \hookrightarrow X \times \mathbb{C}^{n+1}$.

Theorem 8.2.1. *Φ is a bijective correspondence between maps $X \to \mathbb{P}(V)$ and line subbundles of $X \times \mathbb{C}^{n+1}$.*

Proof. Let us define a map Ψ in the other direction, that is,

$$\Psi : \{\text{Line sub-bundles of } X \times \mathbb{C}^{n+1}\} \to \{\text{Smooth maps } X \to \mathbb{P}(V)\}$$

Let $\mathcal{L}$ be a line subbundle of $X \times \mathbb{C}^{n+1}$. We define a map $\Psi(\mathcal{L}) : X \to \mathbb{P}(V)$ by the pointwise description as follows. For $x \in X$, $\Psi(\mathcal{L})(x) = [\mathcal{L}_x]$, where $[\mathcal{L}_x]$ is the class of the line determined by $\mathcal{L}_x$ in $\mathbb{C}^{n+1}$.

Let us check that $\Psi(\mathcal{L})$ is a smooth map. To show this it is enough to show that for each $x \in X$ there is a neighbourhood U of x in X such that $\Psi(\mathcal{L})|_U$ is smooth. So let $x \in X$ and U be a trivializing neighbourhood of x in X. Then we have a non-vanishing smooth section $s \in \Gamma(U, \mathcal{L})$. Let $q : \mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\} \to \mathbb{P}(V)$ be the natural quotient map and $\iota : \mathcal{L} \to X \times \mathbb{C}^{n+1}$ be the inclusion map. Then we have $\Psi(\mathcal{L})|_U = q \circ \pi \circ \iota \circ s$ where $\pi : X \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the projection. Since q, π and ι are smooth, so $\Psi(\mathcal{L})|_U$ is smooth.

We claim $\Psi \circ \Phi = Id$. Let $f : X \to \mathbb{P}^n$ be a smooth map. Let $\mathcal{L}$ denote the line bundle $[f^*\mathcal{O}_{\mathbb{P}(V)}(-1)]$. We claim that the lines $\mathcal{L}_x$ and $\mathcal{O}_{\mathbb{P}(V)}(-1)_{f(x)}$ inside $\mathbb{C}^{n+1}$ are the same. If this claim is true, then it will follow that $\Psi(\mathcal{L})(x) = f(x)$, that is, $\Psi \circ \Phi(f) = f$. However, the claim follows trivially from the canonical identification between the fibers $(f^*E)_x$ and $E_{f(x)}$ for a bundle E over Y and a map $f : X \to Y$. This proves that $\Psi \circ \Phi = Id$.

Next let us show that $\Phi \circ \Psi = Id$. Let us start with a line sub-bundle $\mathcal{L} \hookrightarrow X \times \mathbb{C}^{n+1}$. By the definition of the map f that this defines, it is clear that for each point $x \in X$ the line sub-bundles $\mathcal{L}$ and $f^*\mathcal{O}_{\mathbb{P}(V)}(-1)$ define the same line inside $\mathbb{C}^{n+1}$ over x. Now we easily conclude that these line sub-bundles are the same. For example, consider the short exact sequence

$$0 \to \mathcal{L} \to X \times \mathbb{C}^{n+1} \to \mathcal{Q} \to 0\,,$$

where $\mathcal{Q}$ denotes the quotient. Then $f^*\mathcal{O}_{\mathbb{P}(V)}(-1)$ maps to 0 in $\mathcal{Q}$. This shows that there is a map $f^*\mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{L}$ which is clearly forced to be an isomorphism. This proves that $\Phi \circ \Psi = Id$. This completes the proof of the theorem. $\square$

8.3 Construction of Grassmannian manifold $Gr_k(V)$ and the tautological sub-bundle

Notations :

- $M_{m\times n}(\mathbb{C})$ denotes set of all $m \times n$ matrices with entries from $\mathbb{C}$.
- $\mathscr{I}_k := \{I \subset \{1, 2, \dots, n\} : |I| = k\}$ for $1 \le k \le n$.
- For $A = (A_1, A_2, \dots, A_n) \in M_{k\times n}(\mathbb{C})$ and $I = \{i_1 < i_2 < \cdots < i_k\} \in \mathscr{I}_k$, we define $A_I := (A_{i_1}, A_{i_2}, \dots, A_{i_k}) \in M_{k\times k}(\mathbb{C})$.

Let V be the n dimensional vector space $\mathbb{C}^n$ over $\mathbb{C}$. The grassmannian $Gr_k(V)$ is the space of all k-dimensional subspaces of V. We first construct $Gr_k(V)$ using gluing. Let for each $I \in \mathscr{I}_k$, $U_I \cong \mathbb{C}^{k\times(n-k)}$ be the set

$$U_I = \{A \in M_{k\times n}(\mathbb{C}) \quad | \quad A_I = I_k\}\,.$$

Let for $I, J \in \mathscr{I}_k$, $U_{IJ} \subset U_I$ be the set

$$U_{IJ} := \{A \in U_I \quad | \quad A_J \text{ is non-singular}\}\,.$$

In particular, this means that $U_{II} = U_I$. Define maps $\varphi_{IJ} : U_{IJ} \to U_{JI}$ to be multiplication by $(A_J)^{-1}$, that is,

$$\varphi_{IJ}(A) = (A_J)^{-1} A\,.$$

It is easily checked that these satisfy the cocycle condition, and φ_{IJ}'s are smooth. Clearly gluing these U_I's along φ_{IJ}'s we get $Gr_k(V)$.

Next we will glue to construct a sub-bundle of $Gr_k(V) \times V$ of rank k. First note that over U_I we have a canonical sub-bundle of rank k, given by the following sections $s_{I,1}, s_{I,2}, \ldots, s_{I,k}$ of the bundle $U_I \times V \to U_I$,

$$(a_{mn}) = A \mapsto s_{I,i}(A) = (a_{i1}, a_{i2}, \ldots, a_{in}) \in V \qquad \text{for } 1 \leq i \leq k\,.$$

It is clear that these sections are smooth and since $\text{rank}(A) = k$, so these spans a k dimensional subspace of V . Thus, it defines a sub-bundle of $U_I \times V$ of rank k. Let us denote this sub-bundle by $\mathcal{S}_I$. Now the map multiplication by $(A_J)^{-1}$ fits into a commutative diagram of smooth maps

$$\begin{array}{ccc} \mathcal{S}_I|_{U_{IJ}} & \longrightarrow & \mathcal{S}_J|_{U_{JI}} \\ \downarrow & & \downarrow \\ U_{IJ} & \xrightarrow{\varphi_{IJ}} & U_{JI} \end{array}$$

The top horizontal arrow is given by

$$(A, v_1, \ldots, v_k) \mapsto (\varphi_{IJ}(A), (A_J)^{-1} \cdot v_1, \ldots, (A_J)^{-1} \cdot v_k)\,.$$

As a result we get that the $\mathcal{S}_I$'s glue together to give a sub-bundle of $Gr_k(V) \times V$ of rank k. We call this sub-bundle tautological sub-bundle over $Gr_k(V)$ and denote it by $\mathcal{S}_{Gr_k(V)}$. It has the description that over a point $[w] \in Gr_k(V)$ the fiber is exactly the k dimensional subspace in V corresponding to $[w]$.

It is easily checked that as sets,

$$Gr_k(V) \cong \{A \in M_{n\times k}(\mathbb{C}) : rk(A) = k\}/\sim$$

where $A \sim B$ iff $A = BT$ for some $T \in GL(k, \mathbb{C})$. The natural quotient map $q : \{A \in M_{n\times k}(\mathbb{C}) : rk(A) = k\} \to Gr_k(V)$ is smooth with respect to the given smooth structure on $Gr_k(V)$.

8.4 Universal property of $Gr_k(V)$

Let V be the vector space $\mathbb{C}^n$ over $\mathbb{C}$. On $Gr_k(V)$ we have a short exact sequence of vector bundles, namely,

$$0 \to \mathcal{S}_{Gr_k(V)} \to Gr_k(V) \times V \to \mathcal{Q} \to 0\,.$$

Let $f : X \to Gr_k(V)$ be a smooth map. The pull back of the above short exact sequence to X along f gives a short exact sequence of vector bundles on X

$$0 \to f^*\mathcal{S}_{Gr_k(V)} \to X \times V \to f^*\mathcal{Q} \to 0\,.$$

Thus, consider the map Φ between the following two sets

$$\Phi : \{\text{Smooth maps } X \to Gr_k(V)\} \to \{\text{Sub-bundles of } X \times V \text{of rank } k\}\,,$$

defined as $\Phi(f)$ is the sub-bundle $f^*(\mathcal{S}_{Gr_k(V)}) \hookrightarrow X \times V$.

Theorem 8.4.1. Φ *is a bijective correspondence between maps* $X \to Gr_k(V)$ *and subbundles of* $X \times V$ *of rank* k*.*

Proof. Let us define a map Ψ in the other direction, that is,

$$\Psi : \{\text{Subbundles of } X \times V \text{ of rank } k\} \to \{\text{Smooth maps } X \to Gr_k(V)\}$$

Let $\mathcal{K}$ be a subbundle of $X \times V$. We define a map $\Psi(\mathcal{K}) : X \to Gr_k(V)$ by the pointwise description as follows. For $x \in X$, $\Psi(\mathcal{K})(x) = [\mathcal{K}_x]$, where $[\mathcal{K}_x]$ is the point in $Gr_k(V)$ corresponding to the k dimensional subspace $\mathcal{K}_x$ of V.

Let us check that $\Psi(\mathcal{K})$ is a smooth map. To show this it is enough to show that for each $x \in X$ there is a neighbourhood U of x in X such that $\Psi(\mathcal{K})|_U$ is smooth. So let $x \in X$ and U be a trivializing neighbourhood of x in X i.e. $\mathcal{K}|_U \cong U \times V$.

Then we have k linearly independent smooth sections $s_1, s_2, \ldots, s_k \in \Gamma(U, \mathcal{K})$. For each $y \in U$, viewing $s_i(y)$ as a column vector in V, we have $(s_1(y), \ldots, s_k(y)) \in M_{n\times k}(\mathbb{C})$ of rank k. Let $q : \{A \in M_{n\times k}(\mathbb{C}) : rk(A) = k\} \to Gr_k(V)$ be the natural quotient map and $\pi : X \times V \to V$ be the projection. Then we have $\Psi(\mathcal{K})|_U = q \circ \pi(s_1, \ldots, s_k)$. Since q and π are smooth, so $\Psi(\mathcal{K})|_U$ is smooth.

We claim $\Psi \circ \Phi = Id$. Let $f : X \to Gr_k(V)$ be a smooth map. Let $\mathcal{K}$ denote the bundle $f^*(\mathcal{S}_{Gr_k(V)})$. We claim that the subspaces $\mathcal{K}_x$ and $(\mathcal{S}_{Gr_k(V)})_{f(x)}$ of V are the same. If this claim is true, then it will follow that $\Psi(\mathcal{K})(x) = f(x)$, that is, $\Psi \circ \Phi(f) = f$. However, the claim follows trivially from the canonical identification between the fibers $(f^*E)_x$ and $E_{f(x)}$ for a bundle E over Y and a map $f : X \to Y$. This proves that $\Psi \circ \Phi = Id$.

Next let us show that $\Phi \circ \Psi = Id$. Let us start with a sub-bundle $\mathcal{K} \hookrightarrow X \times V$ of rank k. By the definition of the map f that this defines, it is clear that for each point $x \in X$ the sub-bundles $\mathcal{K}$ and $f^*(\mathcal{S}_{Gr_k(V)})$ define the same subspace of V over x. Now we easily conclude that these sub-bundles are the same. For example, consider the short exact sequence

$$0 \to \mathcal{K} \to X \times V \to \mathcal{N} \to 0\,,$$

where $\mathcal{N}$ denotes the quotient. Then $f^*(\mathcal{S}_{Gr_k(V)})$ maps to 0 in $\mathcal{N}$. This shows that there is a map $f^*(\mathcal{S}_{Gr_k(V)}) \to \mathcal{K}$ which is clearly forced to be an isomorphism. This proves that $\Phi \circ \Psi = Id$. This completes the proof of the theorem. □

9 Some constructions out of vector bundles and their universal properties

9.1 Projectivization of a vector bundle and Universal line bundle

Let M be a manifold and $E \to M$ be a vector bundle over M of rank $n+1$. Let $\{U_i\}_{i \in I}$ be an open cover of M such that E is constructed by gluing $U_i \times \mathbb{C}^{n+1}$ using transition maps $\varphi_{ij} : U_{ij} \to GL(n+1, \mathbb{C})$ where $U_{ij} = U_i \cap U_j$. So for each $x \in M$, $\varphi_{ij}(x)$ induces a map $\widetilde{\varphi}_{ij} : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ given by $\widetilde{\varphi}_{ij}(x)([z]) = [\varphi_{ij}(x)(z)]$ where $z \in \mathbb{C}^{n+1}$. We construct the projective bundle $\mathbb{P}(E)$ over M by gluing $U_i \times \mathbb{P}^n_{\mathbb{C}}$ using smooth maps $U_{ij} \times \mathbb{P}^n_{\mathbb{C}} \to U_{ij} \times \mathbb{P}^n_{\mathbb{C}}$ given by

$$(x, [z]) \mapsto (x, [\varphi_{ij}(x)(z)]) \tag{9.1.1}$$

Let us check that this map is smooth. Assume that $[z] = [z_0 : \ldots : z_n]$ where $z_r \neq 0$. Similarly, let $\varphi_{ij}(x)([z]) = [w_0 : \ldots : w_n]$ where $w_t \neq 0$. Then it is easy to see that $\frac{w_m}{w_t}$ is a smooth function of $\frac{z_n}{z_r}$ and the coordinate functions of $\varphi_{ij}(x)$ in a small neighbourhood of $(x, [z])$.

Let $\pi : \mathbb{P}(E) \to M$ denote the bundle map. We have the bundle π^*E on $\mathbb{P}(E)$. Recall the definition of the pullback. It is constructed by gluing the trivial bundles over the set $\pi^{-1}(U_i)$ using the transition maps $\varphi_{ij} \circ \pi$. Note that $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^n_{\mathbb{C}}$ since E is trivial over U_i. In this particular case, this gluing is given as follows. Glue $\psi_{ij} : U_{ij} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \to U_{ij} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$ using the isomorphism

$$(x, [v], w) \mapsto (x, [\varphi_{ij}(v)], \varphi_{ij}(w))\,.$$

Now on each $U_i \times \mathbb{P}^n_{\mathbb{C}}$ we consider the universal subbundle $U_i \times \mathcal{O}(-1) \subset U_i \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$. We claim that the above gluing ψ_{ij} gives smooth maps $U_{ij} \times \mathcal{O}(-1) \to U_{ij} \times \mathcal{O}(-1)$. Set theoretically this is clear since a point in $U_{ij} \times \mathcal{O}(-1)$ looks like $(x, [v], \lambda v)$. Now we have the following general result. Let $N \subset M$ be an embedded submanifold. Suppose we have a smooth map $X \to M$ such that the image lands inside N, then the set map $X \to N$ is smooth. Applying this we see that there is a commutative square of smooth maps

$$\begin{array}{ccc} U_{ij} \times \mathcal{O}(-1) & \longrightarrow & U_{ij} \times \mathcal{O}(-1) \\ \downarrow & & \downarrow \\ U_{ij} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} & \longrightarrow & U_{ij} \times \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \end{array}$$

This shows that $U_i \times \mathcal{O}(-1)$'s glue together to give a subbundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ of $\pi^* E$ over $\mathbb{P}(E)$. We call $\mathcal{O}_{\mathbb{P}(E)}(-1)$ the universal line bundle.

9.2 Universal property of $\mathbb{P}(E)$

Let E be a vector bundle on a smooth manifold X of rank $n+1$ and $f : Y \to X$ be a smooth map. Let $\pi : \mathbb{P}(E) \to X$ be the projectivization of E and $g : Y \to \mathbb{P}(E)$ be a smooth map satisfying

$$\pi \circ g = f\,. \tag{9.2.1}$$

On $\mathbb{P}(E)$ we have the short exact sequence of vector bundles :

$$0 \to \mathcal{O}_{\mathbb{P}(E)}(-1) \to \pi^*(E) \to \mathcal{Q} \to 0\,.$$

The pull back of this to Y along f gives a short exact sequence of vector bundles on Y

$$0 \to g^*(\mathcal{O}_{\mathbb{P}(E)}(-1)) \to f^* E \to g^* \mathcal{Q} \to 0\,.$$

Thus, consider the map Φ between the following two sets

$$\Phi : \left\{ \begin{array}{c} \text{Smooth maps } Y \xrightarrow{g} \mathbb{P}(E) \\ \text{satisfying } (9.2.1) \end{array} \right\} \to \{\text{Line sub-bundles of } f^*E\},$$

defined as $\Phi(g)$ is the line sub-bundle $g^*(\mathcal{O}_{\mathbb{P}(V)}(-1)) \hookrightarrow f^*E$.

Theorem 9.2.2. *Φ is a bijective correspondence between maps $g : Y \to \mathbb{P}(E)$ satisfying (9.2.1) and line subbundles of f^*E on Y.*

Proof. We have $\widetilde{f} : f^*E \to E$ which makes the following diagram commute

$$\begin{array}{ccc} f^*E & \xrightarrow{\widetilde{f}} & E \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

and fibre-wise the map is given by $\widetilde{f}_y(v) = v$ where $v \in f^*E_y$ which is canonically identified with $E_{f(y)}$. Let us define a map Ψ in the other direction, that is,

$$\Psi : \{\text{Line sub-bundles of } f^*E\} \to \left\{ \begin{array}{c} \text{Smooth maps } Y \xrightarrow{g} \mathbb{P}(E) \\ \text{satisfying } (9.2.1) \end{array} \right\}$$

Let $\mathcal{L}$ be a line subbundle of f^*E. We define a map $\Psi(\mathcal{L}) : Y \to \mathbb{P}(E)$ by the pointwise description as follows. For $y \in Y$, $\Psi(\mathcal{L})(y) = [\widetilde{f}_y(\mathcal{L}_y)]$, where $[\widetilde{f}_y(\mathcal{L}_y)]$ is the point in $\mathbb{P}(E_{f(y)})$ determined by the line $\widetilde{f}_y(\mathcal{L}_y)$ in $E_{f(y)}$.

To check that $\Psi(\mathcal{L})$ is smooth we need only show that for each $y \in Y$ there is a neighbourhood V of y in Y and an open set W in $\mathbb{P}(E)$ containing $\Psi(\mathcal{L})(y)$ such that $\Psi(\mathcal{L})|_V : V \to W$ is smooth. So let $y \in Y$ and U be a E-trivializing neighbourhood of $f(y)$ in X. So we have smooth vector bundle isomorphisms α and β such that

$$\alpha : E|_U \xrightarrow{\sim} U \times \mathbb{C}^{n+1} \quad \text{and} \quad \beta : \mathbb{P}(E)|_U \xrightarrow{\sim} U \times \mathbb{P}^n_{\mathbb{C}}.$$

Now let $V \subset f^{-1}(U)$ be a neighbourhood of y in Y such that $\mathcal{L}$ is trivial over V. So we have a non-vanishing smooth section $s \in \Gamma(V, \mathcal{L})$. Since $\mathcal{L}$ is a sub-bundle of f^*E we view $s(p)$ as a vector in f^*E_p for $p \in U$. So we get $\Psi(\mathcal{L})$ is the composition of the following maps :

$$V \xrightarrow{s} (f^*E)|_V \setminus s_0 \xrightarrow{\widetilde{f}} E|_U \setminus s_0 \xrightarrow{\alpha} U \times (\mathbb{C}^{n+1} \setminus \{(0,\ldots,0)\}) \xrightarrow{(Id,q)} U \times \mathbb{P}^n_{\mathbb{C}} \xrightarrow{\beta^{-1}} \mathbb{P}(E)|_U$$

where s_0 denotes the zero section of the corresponding bundles and $q : \mathbb{C}^{n+1} \setminus \{(0,\ldots,0)\} \to \mathbb{P}^n_{\mathbb{C}}$ is the natural quotient map. Since all maps in this sequence are smooth so $\Psi(\mathcal{L})|_V : V \to \mathbb{P}(E)|_U$ is smooth. And hence $\Psi(\mathcal{L})$ is smooth.

We claim that $\Phi \circ \Psi = Id$. Let $\mathcal{L}$ denote the line bundle $g^*(\mathcal{O}_{\mathbb{P}(E)}(-1))$. By canonical identification of $g^*(\mathcal{O}_{\mathbb{P}(E)}(-1))_y$ and $\mathcal{O}_{\mathbb{P}(E)}(-1)_{g(y)}$, we have that $\mathcal{L}_y$ and $\mathcal{O}_{\mathbb{P}(E)}(-1)_{g(y)}$ are the same inside $\mathbb{C}^{n+1}$. And by definition of $\widetilde{f}$ we have $\widetilde{f}_y(\mathcal{L}_y)$ and $\mathcal{L}_y$ are the same inside $\mathbb{C}^{n+1}$. So $\widetilde{f}_y(\mathcal{L}_y)$ and $\mathcal{O}_{\mathbb{P}(E)}(-1)_{g(y)}$ are same when considered as a subspace of $\mathbb{C}^{n+1}$. So it follows that $\Psi(\mathcal{L})(y) = g(y)$ and hence $\Psi \circ \Phi(g) = g$.

Next we show that $\Phi \circ \Psi = Id$. We start with a line sub-bundle $\mathcal{L} \hookrightarrow f^*E$. By definition of the map $\Psi(\mathcal{L})$ at each point $y \in Y$, the line sub-bundle $\mathcal{L}$ and $\Psi(\mathcal{L})^*\mathcal{O}_{\mathbb{P}(E)}(-1)$ defines the same line inside $\mathbb{C}^{n+1}$ over y. Now we can conclude that these line sub-bundles are the same by considering the short exact sequence

$$0 \to \mathcal{L} \to f^*E \to \mathcal{N} \to 0\,,$$

where $\mathcal{N}$ denotes the quotient. Then $g^*\mathcal{O}_{\mathbb{P}(E)}(-1)$ maps to 0 in $\mathcal{N}$. This shows that there is a map $g^*\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathcal{L}$ which is clearly forced to be an isomorphism. This proves that $\Phi \circ \Psi = Id$. This completes the proof of the theorem. □

9.3 Construction of *Flag* of a vector bundle $\mathcal{F}(E)$

Let M be a manifold and $E \to M$ be a vector bundle over M of rank n. Let $\pi : \mathbb{P}(E) \to M$ be the projectivization of $E \to M$. We have the vector bundle $\pi^*(E)$ over $\mathbb{P}(E)$ of rank n and the universal subbundle S of rank 1. We define the universal quotient bundle Q_1, a vector bundle of rank $n-1$ over $\mathbb{P}(E)$, to be the cokernel of the inclusion $S \hookrightarrow \pi^*E$. Thus, we have a short exact sequence of vector bundles

$$0 \to S \to \pi^*E \to Q_1 \to 0$$

on $\mathbb{P}(E)$. We again apply projectivization on the vector bundle $Q_1 \to \mathbb{P}(E)$ to get the bundle $\pi_1 : \mathbb{P}(Q_1) \to \mathbb{P}(E)$. Similarly, we can get the universal quotient bundle of rank $n-2$, namely, $\pi_1^*Q_1 \to Q_2$, over $\mathbb{P}(Q_1)$. We apply projectivization again to get $\mathbb{P}(Q_2) \to \mathbb{P}(Q_1)$. Repeating this process we get a sequence of bundles of decreasing rank:

$$M \leftarrow \mathbb{P}(E) \leftarrow \mathbb{P}(Q_1) \leftarrow \cdots \leftarrow \mathbb{P}(Q_{n-2}) \leftarrow \mathbb{P}(Q_{n-1})$$

where Q_{i+1} is the universal quotient bundle over $\mathbb{P}(Q_i)$.

We define *flag* of the vector bundle E to be $\mathcal{F}(E) := \mathbb{P}(Q_{n-1})$. A point in $\mathcal{F}(E)$ is of the form $(x, l_1, l_2, \dots, l_n)$ where $x \in M$ and l_1 is a one dimensional subspace of $W_1 := E_x$ and l_{i+1} is a one dimensional subspace of $W_{i+1} := W_i/l_i$ for $i = 1, \dots, n-1$. Let $\eta_i : W_i \to W_{i+1}$ denote the quotient map. Then we get a sequence surjective maps :

$$E_x = W_1 \xrightarrow{\eta_1} W_2 \xrightarrow{\eta_2} W_3 \xrightarrow{\eta_3} \dots \xrightarrow{\eta_{n-1}} W_n \,.$$

Let $V_i := \ker(\eta_i \circ \dots \circ \eta_1)$ is a i-dimensional subspace of E_x for $1 \leq i < n$. So we have a full flag in E_x :

$$V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset V_n = E_x \,.$$

So to each point $(x, l_1, l_2, \dots, l_n)$ in $\mathcal{F}(E)$ we can assign the full flag $\{V_i\}_1^n$ in E_x and clearly this assignment is bijective. So equivalently we can say a point of $\mathcal{F}(E)$ is of the form $(x, V_1, \dots, V_n)$ where $x \in M$ and $V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset V_n = E_x$ is a full flag in E_x.

9.4 The tautological filtration over $\mathcal{F}(E)$

Definition 9.4.1. *Let B be a vector bundle over M of rank n. A full-sequence of sub-bundles $\mathcal{A}$ in B over M is a sequence of sub-bundles $\mathcal{A}$: $A_1 \hookrightarrow A_2 \hookrightarrow \dots \hookrightarrow A_{n-1} \hookrightarrow B$ where A_i is a vector bundle over M of rank i. We denote the quotient bundle of the inclusion $A_i \hookrightarrow B$ by ${}^{B}\!/_{A_i}$.*

Let E be a vector bundle on a smooth manifold X of dimension n and $\mathcal{F}(E) \xrightarrow{\sigma} X$ be *flag* of $E \to X$. We have the sequence of bundles:

$$X \xleftarrow{\pi} \mathbb{P}(E) \xleftarrow{\pi_1} \mathbb{P}(Q_1) \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_{n-2}} \mathbb{P}(Q_{n-2}) \xleftarrow{\pi_{n-1}} \mathbb{P}(Q_{n-1}) = \mathcal{F}(E)$$

where Q_1 is the universal quotient bundle over $\mathbb{P}(E)$, Q_{i+1} is the universal quotient bundle over $\mathbb{P}(Q_i)$ for $i > 0$, π_i's denote the bundle maps as shown in the diagram and $\sigma = \pi \circ \pi_1 \circ \dots \circ \pi_{n-1}$. Let S_1 denotes the universal line bundle over $\mathbb{P}(E)$ and S_{i+1} denotes the universal line bundle over $\mathbb{P}(Q_i)$ for $i > 0$. So we have the following diagram:

$$\begin{array}{ccccccccccc} & & S_1 & & S_2 & & \dots & & S_{n-1} & & \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ X & \xleftarrow[\pi]{} & \mathbb{P}(E) & \xleftarrow[\pi_1]{} & \mathbb{P}(Q_1) & \xleftarrow[\pi_2]{} & \cdots & \xleftarrow[\pi_{n-2}]{} & \mathbb{P}(Q_{n-2}) & \xleftarrow[\pi_{n-1}]{} & \mathbb{P}(Q_{n-1}) = \mathcal{F}(E) \end{array}$$

In particular, by definition we have short exact sequences

$$0 \to S_i \to \pi_{i-1}^* Q_{i-1} \to Q_i \to 0\,.$$

Let us consider the following line bundles over $\mathcal{F}(E)$,

$$B_i := ((\pi_{n-1} \circ \cdots \circ \pi_i)^*(S_i)) \quad 1 \le i < n\,.$$

So B_1 is a line sub-bundle of $E_1 := \sigma^* E$ and B_2 is a line sub-bundle of $E_2 := {E_1}/{B_1}$. Similarly we write $E_{i+1} := {E_i}/{B_i}$ for $1 < i < n$. So B_i is a line sub-bundle of E_i for $1 \le i < n$. Let $\beta_i : E_i \to E_{i+1}$ denote the quotient map, then we have sequence of surjective maps:

$$\sigma^* E = E_1 \xrightarrow{\beta_1} E_2 \xrightarrow{\beta_2} E_3 \xrightarrow{\beta_3} \ldots \xrightarrow{\beta_{n-1}} E_n\,.$$

Hence we get a *full-sequence of sub-bundles* $\mathcal{R}$ *in* $\sigma^* E$ *over* $\mathcal{F}(E)$:

$$\mathcal{R} : \quad R_1 \hookrightarrow R_2 \hookrightarrow \ldots \hookrightarrow R_{n-1} \hookrightarrow \sigma^* E$$

where

$$R_i := \ker(\beta_i \circ \cdots \circ \beta_1)\,.$$

We call $\mathcal{R}$ to be the tautological filtration of vector bundles over $\mathcal{F}(E)$. It has the description that over a point $(x, V_1, \ldots, V_n) \in \mathcal{F}(E)$ the fiber of R_k is exactly the k dimensional subspace V_k of E_x.

9.5 Universal property of $\mathcal{F}(E)$

Let E be a vector bundle on a smooth manifold X of dimension n and $\mathcal{F}(E) \xrightarrow{\sigma} X$ be *flag* of $E \to X$.

Let $f : Y \to X$ be a smooth map and $g : Y \to \mathcal{F}(E)$ be a smooth map which makes the following diagram commute

(9.5.1)
$$\begin{array}{ccc} & & \mathcal{F}(E) \\ & \overset{g}{\nearrow} & \big\downarrow \sigma \\ Y & \xrightarrow{f} & X\,. \end{array}$$

We have the tautological filtration of bundles $\mathcal{R}$ over $\mathcal{F}(E)$. The pullback of $\mathcal{R}$ to Y along g gives a *full-sequence of sub-bundles in* f^*E *over* Y :

$$g^*\mathcal{R} : \quad g^* R_1 \hookrightarrow g^* R_2 \hookrightarrow \ldots \hookrightarrow g^* R_{n-1} \hookrightarrow g^*(\sigma^* E) = f^* E.$$

Consider the map Φ between the following two sets

$$\Phi : \left\{ \begin{array}{c} \text{Smooth maps } Y \xrightarrow{g} \mathcal{F}(E) \\ \text{satisfying } (9.5.1) \end{array} \right\} \to \{\textit{full-sequence of sub-bundles in } f^*E\},$$

defined as: $\Phi(g)$ is the *full-sequence of sub-bundles* $g^*\mathcal{R}$ *in* f^*E.

Theorem 9.5.2. Φ *is bijective.*

Proof. Let us define a map Ψ in the other direction, that is,

$$\Psi : \{\textit{full-sequence of sub-bundles in } f^*E\} \to \left\{ \begin{array}{c} \text{Smooth maps } Y \xrightarrow{g} \mathcal{F}(E) \\ \text{satisfying } (9.5.1) \end{array} \right\}$$

Let

$$\mathcal{A} : \quad A_1 \hookrightarrow A_2 \hookrightarrow \ldots \hookrightarrow A_{n-1} \hookrightarrow f^*E$$

be a *full-sequence of sub-bundles* in f^*E over Y. So A_1 is a line sub-bundle of f^*E over Y. By theorem 9.2.2 we get a unique smooth map $g_1 : Y \to \mathbb{P}(E)$ which satisfies $g_1^*(S_1) = g_1^*(\mathcal{O}_{\mathbb{P}(E)}(-1)) = A_1$. So $g_1^*(Q_1) = f^*E/A_1$. Now we have the following commutative diagram :

$$\begin{array}{ccc} f^*E/A_1 = g_1^*Q_1 & \xrightarrow{\widetilde{g_1}} & Q_1 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g_1} & \mathbb{P}(E) \end{array}$$

and A_2/A_1 is a line sub-bundle of f^*E/A_1 over Y. So again by theorem 9.2.2 we get a unique smooth map $g_2 : Y \to \mathbb{P}(Q_1)$ satisfying $g_2^*(S_2) = A_2/A_1$. So $g_2^*(Q_2) = f^*E/A_2$. So again we get the commutative diagram :

$$\begin{array}{ccc} f^*E/A_2 = g_2^*Q_2 & \xrightarrow{\widetilde{g_2}} & Q_2 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g_2} & \mathbb{P}(Q_1) \end{array}$$

and A_3/A_2 is a line sub-bundle of f^*E/A_2 over Y. Proceeding in a similar way we will get a unique smooth map $g_n : Y \to \mathbb{P}(Q_{n-1}) = \mathcal{F}(E)$. We define $\Psi(\mathcal{A}) := g_n$. It is clear from the construction that $\Phi \circ \Psi = Id$.

We show that Φ is injective. Let g and h be two smooth maps : $Y \to \mathcal{F}(E)$ satisfying (9.5.1) such that $\Phi(g) = \Phi(h)$. Then in particular two line bundles g^*R_1 and h^*R_1 over Y are same which equivalent to saying that the bundles $(\pi_1 \circ \cdots \circ \pi_{n-1} \circ g)^*(S_1)$ and $(\pi_1 \circ \cdots \circ \pi_{n-1} \circ h)^*(S_1)$ are same. So by 9.2.2 we have $\pi_1 \circ \cdots \circ \pi_{n-1} \circ g = \pi_1 \circ \cdots \circ \pi_{n-1} \circ h = f_1$ say. So now we have the commutative diagram,

$$\begin{array}{ccc} & & \mathbb{P}(Q_1) \\ & \overset{g_1, h_1}{\nearrow} & \downarrow{\scriptstyle \pi_1} \\ Y & \xrightarrow{f_1} & \mathbb{P}(E)\,. \end{array}$$

where $g_1 := \pi_2 \circ \cdots \circ \pi_{n-1} \circ g$ and $h_1 := \pi_2 \circ \cdots \circ \pi_{n-1} \circ h$.

Now the line sub-bundles g^*R_2/g^*R_1 and h^*R_2/h^*R_1 of f^*E/g^*R_1 over Y are same which is equivalent to saying that the sub-bundles $g_1^*(S_2)$ and $h_1^*(S_2)$ of $f_1^*Q_1$ are same. Again using 9.2.2 we get that $\pi_2 \circ \cdots \circ \pi_{n-1} \circ g = \pi_2 \circ \cdots \circ \pi_{n-1} \circ h$. Proceeding similarly after n steps we get that $g = h$.

So Φ is both injective and surjective. Hence Φ is a bijective correspondence and the theorem is proved. □

9.6 Poincare polynomial of the flag of a bundle

For a vector bundle $E \to M$ of rank n we have the sequence of bundles

$$\mathcal{F}(E) = \mathbb{P}(Q_{n-1}) \to \mathbb{P}(Q_{n-2}) \to \cdots \to \mathbb{P}(Q_1) \to \mathbb{P}(E) \to M$$

where Q_{i+1} is the universal quotient bundle over $\mathbb{P}(Q_i)$. We have proved in Lecture 21 that the poincare polynomial of $\mathbb{P}(E)$ is given by,

$$P(\mathbb{P}(E), t) = P(M, t)\frac{1 - t^{2n}}{1 - t^2}\,.$$

Again

$$\begin{aligned} P(\mathbb{P}(Q_1), t) &= P(\mathbb{P}(E), t)\frac{1 - t^{2(n-1)}}{1 - t^2} \\ &= P(M, t)\frac{1 - t^{2n}}{1 - t^2} \cdot \frac{1 - t^{2(n-1)}}{1 - t^2}\,. \end{aligned}$$

So by induction we get,

$$P(\mathbb{P}(Q_{n-1}),t) = P(M,t)\cdot \frac{\prod_{i=2}^{n}(1-t^{2i})}{(1-t^2)^{n-1}}\,.$$

So poincare polynomial of $\mathcal{F}(E)$ is given by,

$$P(\mathcal{F}(E),t) = P(M,t)\cdot \frac{\prod_{i=2}^{n}(1-t^{2i})}{(1-t^2)^{n-1}}\,. \tag{9.6.1}$$

10 De Rham Cohomology of Complex Grassmannian manifold

We will determine the poincare polynomial of grassmannian manifold. Let $Gr_k(\mathbb{C}^n)$ denotes the complex grassmannian manifold : the set of all k dimensional linear subspaces of $\mathbb{C}^n$. Let us consider the trivial bundle $F = Gr_k(\mathbb{C}^n) \times \mathbb{C}^n$ over $Gr_k(\mathbb{C}^n)$. Recall that we have the universal subbundle $\mathcal{S}_{Gr_k(\mathbb{C}^n)}$ of F whose fiber at a point $[V] \in Gr_k(\mathbb{C}^n)$ is the the k dimensional subspace V. We write the vector bundle $\mathcal{S}_{Gr_k(\mathbb{C}^n)}$ simply as S here. We define the universal quotient bundle Q over $Gr_k(\mathbb{C}^n)$ to be the cokernel of the inclusion : $S \hookrightarrow F$. Now we take the *flag* of S to get a bundle $\sigma : \mathcal{F}(S) \to Gr_k(\mathbb{C}^n)$. Pulling back Q we get a vector bundle $\sigma^*(Q)$ over $\mathcal{F}(S)$ of rank $n-k$. Again taking *flag* of $\sigma^*(Q)$ we get a sequence of bundle

$$\mathcal{F}(\sigma^*Q) \xrightarrow{\tau} \mathcal{F}(S) \xrightarrow{\sigma} Gr_k(\mathbb{C}^n).$$

Now considering $\mathbb{C}^n$ as a vector bundle E over a point $\{*\}$ we construct $\mathcal{F}(E) \xrightarrow{\pi} \{*\}$.
Claim : $\mathcal{F}(E)$ and $\mathcal{F}(\sigma^*Q)$ are diffeomorphic as smooth manifolds.

Proof. Let us first give a map $\delta : \mathcal{F}(\sigma^*Q) \to \mathcal{F}(E)$. Using the universal property of $\mathcal{F}(E)$ it suffices to give a full sequence of sub-bundles of the trivial bundle of rank n over $\mathcal{F}(\sigma^*Q)$. On $Gr_k(\mathbb{C}^n)$ there is a short exact sequence

$$0 \to S \to F \to Q \to 0\,.$$

Pulling this back along $\sigma \circ \tau$ we get the following short exact sequence on $\mathcal{F}(\sigma^*Q)$

$$0 \to (\sigma\circ\tau)^*S \to (\sigma\circ\tau)^*F \xrightarrow{\theta} (\sigma\circ\tau)^*Q \to 0\,.$$

Since $(\sigma\circ\tau)^*Q \cong \tau^*(\sigma^*Q)$, there is a canonical full sequence of sub-bundles in $\tau^*(\sigma^*Q)$ over the space $\mathcal{F}(\sigma^*Q)$. We can take the inverse image of this sequence under the map θ, let us call this sequence of sub-bundles $\mathcal{R}'$. This is not a full sequence in the trivial bundle. We also have a full sequence of sub-bundles in σ^*S over the space $\mathcal{F}(S)$. Let us call this sequence $\mathcal{R}$. Then it is clear that $\tau^*\mathcal{R} \subset \mathcal{R}'$ is a full sequence of sub-bundles of the trivial bundle $(\sigma\circ\tau)^*F$ over $\mathcal{F}(\sigma^*Q)$. By the universal property of $\mathcal{F}(E)$ we get a map $\delta : \mathcal{F}(\sigma^*Q) \to \mathcal{F}(E)$.

We claim that δ is bijective. This will become clear from the set theoretic description of δ. Let $p \in \mathcal{F}(\sigma^*Q)$. Then we get a point $\sigma\circ\tau(p) \in Gr_k(\mathbb{C}^n)$, which corresponds to a k-dimensional subspace $T_k \subset \mathbb{C}^n$. The fiber of S over the point $\sigma \circ \tau(p)$ is precisely T_k. The point $\tau(p)$ corresponds to a full flag inside T_k. The point p corresponds to a full flag inside $\mathbb{C}^n/T_k$. But these two full flags give rise to a full flag inside $\mathbb{C}^n$. This corresponds to a point in $\mathcal{F}(E)$, which is precisely $\delta(p)$. There is a bijection between the two sets

1. A full flag in $\mathbb{C}^n$
2. A k-dimensional subspace $T_k \subset \mathbb{C}^n$, a full flag inside T_k and a full flag inside $\mathbb{C}^n/T_k$.

From this bijection and the set theoretic description of δ it is clear that δ is bijective.

Finally let us construct a map which is a smooth inverse to δ. Firstly to get a map $\mathcal{F}(E) \to Gr_k(\mathbb{C}^n)$, by universal property of Grassmannians 8.4.1 it is enough to get a subbundle of trivial bundle of rank k over $\mathcal{F}(E)$.

Recall that over $\mathcal{F}(E)$ we have the tautological filtration of bundles $\mathcal{R} : R_1 \hookrightarrow R_2 \hookrightarrow \ldots \hookrightarrow R_{n-1} \hookrightarrow \pi^*E$. In particular R_k is a subbundle of trivial bundle π^*E of rank k over $\mathcal{F}(E)$. So we get a smooth map $h : \mathcal{F}(E) \to Gr_k(\mathbb{C}^n)$.

Now $\mathcal{R}_k : R_1 \hookrightarrow R_2 \hookrightarrow \ldots \hookrightarrow R_k = h^*S$ is a *full-sequence* of sub-bundles in h^*S over $\mathcal{F}(E)$. So by theorem 9.5.2 we get a smooth map

$$g : \mathcal{F}(E) \to \mathcal{F}(S)$$

satisfying $\sigma \circ g = h$ which has the set theoretic description,

$$(V_1, V_2, \ldots, V_n) \mapsto ([V_k], V_1, \ldots, V_k).$$

Now $g^*(\sigma^*Q) = (\sigma \circ g)^*Q = h^*Q$ and $\mathcal{R}/R_k : R_{k+1}/R_k \hookrightarrow R_{k+2}/R_k \hookrightarrow \ldots \hookrightarrow R_n/R_k = h^*Q$ is a *full-sequence* of sub-bundles in h^*Q over $\mathcal{F}(E)$. So again using theorem 9.5.2 we get a smooth map

$$\psi : \mathcal{F}(E) \to \mathcal{F}(\sigma^*Q)$$

which has the set theoretic description,

$$(V_1, V_2, \ldots, V_n) \mapsto ([V_k], V_1, \ldots, V_k, V_{k+1}/V_k, \ldots, V_n/V_k)\,.$$

By set theoretic description of δ and ψ it is clear that ψ is smooth inverse of δ. So we conclude that δ is a diffeomorphism and hence $\mathcal{F}(E)$ and $\mathcal{F}(\sigma^*Q)$ are diffeomorphic as smooth manifold. $\square$

Now from equation (9.6.1) we have,

$$P(\mathcal{F}(E),t) = P(*,t)\cdot\frac{\prod_{i=1}^{n}(1-t^{2i})}{(1-t^2)^n} = \frac{\prod_{i=1}^{n}(1-t^{2i})}{(1-t^2)^n}\,. \tag{10.0.1}$$

Again

$$\begin{aligned} P(\mathcal{F}(\sigma^*Q),t) &= P(\mathcal{F}(S),t)\cdot\frac{\prod_{i=1}^{n-k}(1-t^{2i})}{(1-t^2)^{n-k}} \\ &= P(Gr_k(\mathbb{C}^n),t)\cdot\frac{\prod_{i=1}^{k}(1-t^{2i})}{(1-t^2)^k}\cdot\frac{\prod_{i=1}^{n-k}(1-t^{2i})}{(1-t^2)^{n-k}} \qquad (10.0.2)\end{aligned}$$

By the claim we have $\mathcal{F}(E)\cong\mathcal{F}(\sigma^*Q)$, so combining equation 10.0.1 and 10.0.2 we get,

$$P(Gr_k(\mathbb{C}^n),t) = \frac{\prod_{i=1}^{n}(1-t^{2i})}{\prod_{i=1}^{k}(1-t^{2i})\prod_{i=1}^{n-k}(1-t^{2i})}\,.$$

11 De Rham cohomology of $U(n)$

The unitary group $U(n)$ is the group of all $n \times n$ unitary matrices which is a compact submanifold of the smooth manifold $GL(n, \mathbb{C})$.

Proposition 11.0.1. *For $n > 1$, We have a locally trivial fiber bundle :*

$$U(n-1) \hookrightarrow U(n) \to S^{2n-1} . \tag{11.0.2}$$

Proof. We consider the map

$$f : U(n) \to S^{2n-1}$$

$$A \mapsto Ae_1$$

where $\{e_1, \ldots, e_n\}$ is the standard ordered basis of $\mathbb{C}^n$ over $\mathbb{C}$. Clearly f is smooth and surjective. Since $U(n)$ and S^{2n-1} are compact and hausdorff, so f is proper. Clearly fiber over each point in S^{2n-1} is diffeomorphic to the closed submanifold $U(n-1)$ of $U(n)$. So if we show that f is a submersion and since the fiber is compact it is easy to see that we get a locally trivial fiber bundle (11.0.2).

To see that f is submersion : Firstly we show f is submersion at identity $I_n \in U(n)$. We have

$$Df_{I_n} : T_{I_n}U(n) \to T_{e_1}S^{2n-1} .$$

Considering $U(n)$ as a submanifold of $GL(n, \mathbb{C})$ and S^{2n-1} as a submanifold of $\mathbb{C}^n$ we clearly have,

$$Df_{I_n} : \{A \in GL(n, \mathbb{C}) : A \text{ is skew hermition}\} \to \{x \in \mathbb{C}^n : x^T e_1 = 0\}$$

given by,

$$Df_{I_n}(B) = Be_1 .$$

If $x \in T_{e_1}S^{2n-1}$, so we have $x^T e_1 = 0$. Writing $x = \sum_{i=1}^n x_i e_i$ with $x_i \in \mathbb{C}$, we have $x_1 = 0$. So taking

$$B = \begin{bmatrix} 0 & \overline{x_2} & \dots & \overline{x_n} \\ x_2 & 0 & \dots & 0 \\ \vdots & & \ddots & \\ x_n & 0 & \dots & 0 \end{bmatrix}$$

clearly B is skew hermition and $Be_1 = x$. This proves that Df_{I_n} is surjective and hence f is a submersion at I_n.

Now to see that f is a submersion at any $U \in U(n)$, we observe that the following diagram is commutative :

$$\begin{array}{ccc} U(n) & \xrightarrow{L_U} & U(n) \\ \downarrow f & & \downarrow f \\ S^{2n-1} & \xrightarrow{\phi_U} & S^{2n-1} \end{array}$$

where L_U is left multiplication by U i.e. $L_U(A) = UA$ and $\phi_U(x) = Ux$. Moreover both L_U and ϕ_U are diffeomorphisms. So from the commutative diagram

$$\begin{array}{ccc} T_{I_n}U(n) & \xrightarrow{(DL_U)_{I_n}} & T_U U(n) \\ \downarrow Df_{I_n} & & \downarrow Df_U \\ T_{e_1}S^{2n-1} & \xrightarrow{(D\phi_U)_{e_1}} & T_{(Ue_1)}S^{2n-1} \end{array}$$

it is clear that f is submersion at $U \in U(n)$. $\square$

Before going to cohomology of $U(n)$, we first see some definitions and theorems to be used in the computation of cohomology.

Definition 11.0.3 (Exterior Algebras). *The exterior algebra $\Lambda[a_1, a_2, \ldots, a_r]$ over $\mathbb{R}$ is the free $\mathbb{R}$-module with basis the finite products $a_{i_1} \cdots a_{i_k}$ for $i_1 < \cdots < i_k$ with multiplication defined by the rules $a_i a_j = -a_j a_i$ (in particular, this implies $a_i^2 = 0$). Defining the empty product of a_i's to be $1 \in \mathbb{R}$, $\Lambda[a_1, a_2, \ldots, a_r]$ becomes an algebra with the identity element 1.*

We can make $\Lambda[a_1, a_2, \ldots, a_r]$ a graded algebra just by defining degree of the elements $a_1, \ldots, a_r$. For example if we have degree$(a_i) = d_i \geq 0$ for $i = 1, 2, \ldots, r$, then degree of the monomial $a_{i_1} \cdots a_{i_k}$ is $d_{i_1} + \cdots + d_{i_k}$ for $i_1 < \cdots < i_k$ and $k \leq r$.

Theorem 11.0.4 (Special case of Leray-Serre spectral sequence). *Let $F \hookrightarrow E \to B$ be a locally trivial fiber bundle and suppose F is connected and B is simply connected. Then there is a first quadrant cohomological spectral sequence of algebras $\{E_r^{*,*}, d_r\}$, where d_r is of bidegre $(r, 1-r)$, such that $E_2^{p,q} \cong H^p(B, \mathbb{R}) \otimes H^q(F, \mathbb{R})$ and the spectral sequence converges to $H^*(E, \mathbb{R})$ as an algebra. Moreover, the differential d_r is an antiderivation i.e.*

$$d_r(a \cdot b) = d(a) \cdot b + (-1)^{u+v} a \cdot d(b) \quad \textit{where} \quad a \in E_r^{u,v}.$$

Now we compute the De Rham cohomology of $U(n)$:

Proposition 11.0.5. *For $n \geq 1$ the cohomology ring of $U(n)$ is given by*

$$H^*(U(n);\mathbb{R}) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}]$$

isomorphic as graded algebra with degree$(x_i) = i$.

Proof. For $n = 1$, clearly $U(1) = \{a \in C^* : \quad a\overline{a} = 1 = \overline{a}a\} = S^1$. So

$$H^i(U(1);\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Letting x_1 to be a generator of $H^1(U(1))$, we have $x_1^2 = 0$. So

$$H^*(U(1);\mathbb{R}) \cong \Lambda[x_1] \quad \text{with degree}(x_1) = 1\,.$$

We proceed by induction on n. So we assume the statement to be true for $n = m$. So we have

$$H^*(U(m);\mathbb{R}) \cong \Lambda[x_1, x_3, \ldots, x_{2m-1}] \quad \text{with degree}(x_i) = i\,.$$

So by graded ring structure of $H^*(U(m);\mathbb{R})$, we have $x_i \in H^i(U(m);\mathbb{R})$. Now we prove the statement for $n = m + 1$.

Considering the fiber bundle (11.0.2) for $n = m + 1$, we have the locally trivial fiber bundle

$$U(m) \hookrightarrow U(m+1) \to S^{2m+1}\,.$$

with $U(m)$ connected and S^{2m+1} simply connected. So using theorem (11.0.4) we will get a Leray-Serre spectral sequnece of algebras $\{E_r^{*,*}, d_r\}$ where

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1} \quad \text{and} \quad E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{Im} d_r^{p-r,q+r-1}$$

and $E_2^{p,q} \cong H^p(S^{2m+1};\mathbb{R}) \otimes H^q(U(m);\mathbb{R})$ and the spectral sequence converges to $H^*(U(m+1);\mathbb{R})$ as an algebra.

By construction of spectral sequence we have

$$E_2^{*,*} \cong H^*(S^{2m+1};\mathbb{R}) \otimes H^*(U(m);\mathbb{R})$$

isomorphic as graded algebras.

Let x_{2m+1} be a generator of $H^{2m+1}(S^{2m+1};\mathbb{R})$ so that $H^*(S^{2m+1};\mathbb{R}) \cong \Lambda[x_{2m+1}]$ isomorphic as graded algebra where degree of x_{2m+1} is $2m+1$. So

$$E_2^{*,*} \cong \Lambda[x_1, x_3, \ldots, x_{2m-1}] \otimes \Lambda[x_{2m+1}] \cong \Lambda[x_1, x_3, \ldots, x_{2m+1}]$$

isomorphic as graded algebras and degree of x_i is i.

Claim : For $r \geq 2$,

$$E_r^{*,*} \cong H^*(S^{2m+1};\mathbb{R}) \otimes H^*(U(m);\mathbb{R}) \cong \Lambda[x_1, x_3, \ldots, x_{2m+1}]$$

isomorphic as graded algebras and $E_r^{p,q} \cong H^p(S^{2m+1};\mathbb{R}) \otimes H^q(U(m);\mathbb{R})$.

Proof : We have the result for $r = 2$. We proceed by induction on r. So we assume we have the result for $r = s \geq 2$ i.e.

$$E_s^{*,*} \cong H^*(S^{2m+1};\mathbb{R}) \otimes H^*(U(m);\mathbb{R}) \cong \Lambda[x_1, x_3, \ldots, x_{2m+1}]$$

isomorphic as graded algebras. We need to prove the result for $r = s+1$. We intend to prove that $d_s = 0$. Then it will follow that $E_{s+1}^{*,*} \cong E_s^{*,*}$ isomorphic as graded algebras. So the claim is proved by induction.

To show that $d_s = 0$: Since d_s is an antiderivation, it is enough to show that d_s takes the generators to 0 i.e. $d_s(x_i) = 0$ for $i = 1, 3, \ldots, 2m+1$.

- For $i = 1, 3, \ldots, 2m-1$,

$$x_i \in E_s^{0,i} \quad \text{and} \quad d_s^{0,i} : E_s^{0,i} \to E_s^{s,i-s+1}.$$

  - If $s \neq 2m+1$, $H^s(S^{2m+1};\mathbb{R}) = 0$ and hence

$$E_s^{s,i-s+1} \cong H^s(S^{2m+1};\mathbb{R}) \otimes H^{i-s+1}(U(m);\mathbb{R}) \cong 0$$

    So $d_s(x_i) = 0$
  - If $s = 2m+1$, then $i - s + 1 = i - 2m < 0$. So $E_s^{s,i-s+1} \cong 0$ and hence $d_s(x_i) = 0$.

- For $i = 2m+1$,

$$x_{2m+1} \in E_s^{2m+1,0} \quad \text{and} \quad d_s^{2m+1,0} : E_s^{2m+1,0} \to E_s^{2m+1+s,1-s}.$$

  Now $1 - s < 0$, so $E_s^{2m+1+s,1-s} = 0$ and hence $d_s(x_{2m+1}) = 0$.

So we have $d_s(x_i) = 0$ for $i = 1, 3, \ldots, 2m+1$ and hence $d_s = 0$. So we have proved that for $r \geq 2$, $E_r^{*,*} \cong \Lambda[x_1, x_3, \ldots, x_{2m+1}]$ as graded algebras.

So

$$H^*(U(m+1); \mathbb{R}) \cong E_\infty^{*,*} \cong \Lambda[x_1, x_3, \ldots, x_{2m+1}] .$$

isomorphic as graded algebra with degree$(x_i) = i$.

So by induction we have proved that for $n \geq 1$,

$$H^*(U(n); \mathbb{R}) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}]$$

isomorphic as graded algebra with degree$(x_i) = i$.

□

12 De Rham cohomology of $GL(n, \mathbb{C})$

Lemma 12.0.1. *$GL(n, \mathbb{C})$ is diffeomorphic to $U(n) \times Herm(n, \mathbb{C})$ where $U(n)$ denotes the set of all $n \times n$ complex unitary matrices and $Herm(n, \mathbb{C})$ denotes the set of all $n \times n$ complex hermitian matrices.*

Proof. We will prove the Lemma in 3 steps.

Step 1. We will show that the set of all $n \times n$ complex positive definite hermitian matrices $Herm^+(n, \mathbb{C})$ is open in $Herm(n, \mathbb{C})$. Hence, it is a manifold of dimension n^2. We show that the complement of $Herm^+(n, \mathbb{C})$ in $Herm(n, \mathbb{C})$ is closed. Let $\{A_m\}$ be a sequence in $Herm(n, \mathbb{C}) \backslash Herm^+(n, \mathbb{C})$ such that $\{A_m\}$ converges to $A \in Herm(n, \mathbb{C})$. So for each $m \in \mathbb{N}$, $\exists x_m \in \mathbb{C}^n \setminus \{0\}$ s.t. $x_m^* A_m x_m \leq 0$. Replacing x_m with $\frac{x_m}{||x_m||}$ we can assume $x_m \in S^{2n-1}$ for all $m \in \mathbb{N}$. Since S^{2n-1} is compact, so $\{x_m\}$ has a convergent subsequence say $\{x_{m_k}\}$ and let $\lim_{k \to \infty} x_{m_k} = x \in S^{2n-1}$. Then $x^* A x = \lim_{k \to \infty}(x_{m_k}^* A_{m_k} x_{m_k}) \leq 0$. So $A \in Herm(n, \mathbb{C}) \setminus Herm^+(n, \mathbb{C})$. So complement of $Herm^+(n, \mathbb{C})$ is closed in $Herm(n, \mathbb{C})$ and hence $Herm^+(n, \mathbb{C})$ is open in $Herm(n, \mathbb{C})$. Hence $Herm^+(n, \mathbb{C})$ is a manifold of dimension n^2.

Step 2. In this step we shall show that $GL(n, \mathbb{C})$ is diffeomorphic to $U(n) \times$ $Herm^+(n, \mathbb{C})$. If $P \in Herm^+(n, \mathbb{C})$ then $\exists$ an complex unitary matrix Q such that QPQ^{-1} is diagonal. Say $QPQ^{-1} = D = (d_{ij})$ with $0 < d_{11} \leq d_{22} \leq \cdots \leq d_{nn}$ and $d_{ij} = 0$ for $i \neq j$. Then define $\sqrt{P} = Q^{-1}(\text{diag}(\sqrt{d_{11}}, \sqrt{d_{22}}, \ldots, \sqrt{d_{nn}}))Q$ where $\sqrt{d_{ii}}$ is positive square root of d_{ii}. We show that $\sqrt{P}$ is well defined. Say R be another complex unitary matrix such that $RPR^{-1} = D$. Then we have, $Q^{-1}DQ = R^{-1}DR$. Letting $(b_{ij}) = B = RQ^{-1}$ we get $BD = DB$. Hence for any i, j,

$$\sum_{k=1}^{n} b_{ik}d_{kj} = \sum_{k=1}^{n} d_{ik}b_{kj}$$

which implies, $b_{ij}d_{jj} = d_{ii}b_{ij}$. So $b_{ij}(d_{jj} - d_{ii}) = 0$. Since $\sqrt{d_{ii}}, \sqrt{d_{jj}} > 0$, we get $b_{ij}(\sqrt{d_{jj}} - \sqrt{d_{ii}}) = 0$. So $b_{ij}\sqrt{d_{jj}} = \sqrt{d_{ii}}b_{ij}$. This implies that

$$\sum_{k=1}^{n} b_{ik}\sqrt{d_{kj}} = \sum_{k=1}^{n} \sqrt{d_{ik}}b_{kj}$$

So

$$B\begin{pmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{pmatrix} = \begin{pmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{pmatrix} B$$

So

$$Q^{-1}(\mathrm{diag}(\sqrt{d_{11}}, \sqrt{d_{22}}, \dots, \sqrt{d_{nn}}))Q = R^{-1}(\mathrm{diag}(\sqrt{d_{11}}, \sqrt{d_{22}}, \dots, \sqrt{d_{nn}}))R$$

Hence $\sqrt{P}$ is well-defined and $\sqrt{P} \in Herm^{+}(n, \mathbb{C})$.
Define

$$\begin{aligned} \Phi : GL(n, \mathbb{C}) &\to U(n) \times Herm^{+}(n, \mathbb{C}) \\ A &\mapsto (A(\sqrt{A^*A})^{-1}, \sqrt{A^*A}) \end{aligned}$$

And

$$\begin{aligned} \Psi : U(n) \times Herm^{+}(n, \mathbb{C}) &\to GL(n, \mathbb{C}) \\ (A, B) &\mapsto AB \end{aligned}$$

Clearly Ψ and Φ are inverses of each other and Ψ is smooth. To show that Φ is smooth it is enough to show that the map $P \mapsto \sqrt{P}$ is smooth. Let $f, g : Herm^{+}(n, \mathbb{C}) \to Herm^{+}(n, \mathbb{C})$ defined by $f(P) = P^2$ and $g(P) = \sqrt{P}$. Then f is smooth and f and g are inverses of each other. Let $P \in Herm^{+}(n, \mathbb{C})$ then we have, $Df_P : Herm(n, \mathbb{C}) \to Herm(n, \mathbb{C})$ given by $Df_P(X) = PX +$

XP. We show that Df_P is an isomorphism. If $PX + XP = 0$ for some $X \in Herm(n, \mathbb{C})$, then $PXP^{-1} = -X$. Now $\exists$ complex unitary matrix Q such that $QPQ^{-1} = D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i > 0$. Letting $(y_{ij}) = Y = QXQ^{-1}$, we have $DYD^{-1} = -Y$. So $\lambda_i y_{ij} \lambda_j^{-1} = -y_{ij}$. Since $\frac{\lambda_i}{\lambda_j} > 0$ so $y_{ij} = 0$ and hence X= O.

So by inverse function theorem f is diffeomorphism. Hence g is smooth. So Φ is smooth. Hence Ψ is a diffeomorphism.

Step 3. In this step we shall show that $Herm^+(n, \mathbb{C})$ is diffeomorphic to $Herm(n, \mathbb{C})$. We take the map

$$\exp : Herm(n, \mathbb{C}) \to Herm^+(n, \mathbb{C})$$
$$A \mapsto \exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

We prove that this is a diffeomorphism. We know exp is smooth. Now $\exp(O) = I_n$ and $D\exp_0$ is the identity map hence isomorphism. By inverse function theorem we get open sets $U \subset Herm(n, \mathbb{C})$ containing O and $V \subset Herm^+(n, \mathbb{C})$ containing I_n such that $\exp : U \to V$ is diffeomorphism. Say $\varphi : V \to U$ be the smooth inverse of exp.

We construct a global inverse of exp. We have already proved that taking square root is a smooth map on positive definite hermitian matrices. Let $B \in Herm^+(n, \mathbb{C})$. Then $\lim_{k\to\infty} B^{1/2^k} = I_n$. So $\exists k \in \mathbb{N}$ such that $B^{1/2^k} \in V$. Define

$$\log : Herm^+(n, \mathbb{C}) \to Herm(n, \mathbb{C}) \quad \text{by} \quad \log(B) = 2^k \varphi(B^{1/2^k})$$

log is well defined : Say if $m, k \in \mathbb{N}$ such that $B^{1/2^m}, B^{1/2^k} \in V$. Then $\varphi(B^{1/2^m}), \varphi(B^{1/2^k}) \in U$. Now

$$\begin{aligned}
\exp\left(\frac{1}{2^k}\varphi(B^{1/2^m})\right) &= (\exp(\varphi(B^{1/2^m}))^{\frac{1}{2^k}} \\
&= B^{\frac{1}{2^{(k+m)}}} \\
&= (\exp(\varphi(B^{1/2^k}))^{\frac{1}{2^m}} \\
&= \exp\left(\frac{1}{2^m}\varphi(B^{1/2^k})\right).
\end{aligned}$$

Since exp is a diffeomorphism on U, so $\frac{1}{2^k}\varphi(B^{1/2^m}) = \frac{1}{2^m}\varphi(B^{1/2^k})$, that is, $2^m\varphi(B^{1/2^m}) = 2^k\varphi(B^{1/2^k})$. So log is well defined.

Now for $B \in Herm^+(n,\mathbb{C})$, if $B^{1/2^k} \in V$,

$$\exp(\log(B)) = \exp(2^k\varphi(B^{1/2^k}) = (\exp(\varphi(B^{1/2^k})))^{2^k} = (B^{1/2^k})^{2^k} = B.$$

For $A \in Herm(n,\mathbb{C})$, $\exists m \in \mathbb{N}$ such that $\frac{1}{2^m}A \in U$. Then $(\exp A)^{1/2^m} = \exp(\frac{1}{2^m}A) \in V$. Now

$$\log(\exp(A)) = 2^m\varphi(\exp(\frac{1}{2^m}A)) = 2^m\frac{1}{2^m}A = A.$$

So, log is inverse of exp. Since φ is smooth, so log is also smooth. Hence exp is a diffeomorphism.

Combining steps (2) and (3) we get $GL(n,\mathbb{C})$ is diffeomorphic to $U(n) \times Herm(n,\mathbb{C})$. $\square$

Since, $Herm(n,\mathbb{C})$ is diffeomorphic to $\mathbb{R}^{n^2}$. So, $H^*(Herm(n,\mathbb{C})) \cong H^*(\mathbb{R}^{n^2})$. By Kunneth formula we will get $H^*(GL(n,\mathbb{C})) \cong H^*(U(n))$. So using proposition (11.0.5) we have

$$H^*(GL(n,\mathbb{C})) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}]$$

isomorphic as graded algebra with degree$(x_i) = i$.

13 De Rham cohomology of $SU(n)$

The special unitary group $SU(n)$ is the group of all $n \times n$ unitary matrices with determinant 1 which is a compact submanifold of the smooth manifold $GL(n, \mathbb{C})$. Clearly $SU(1)$ is a manifold with a single element. So

$$H^i(SU(1); \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \tag{13.0.1}$$

Proposition 13.0.2. *For $n > 1$ the cohomology ring of $SU(n)$ is given by*

$$H^*(SU(n); \mathbb{R}) \cong \Lambda[x_3, x_5, \ldots, x_{2n-1}]$$

isomorphic as graded algebras with degree$(x_i) = i$.

Proof. For $n = 2$, $SU(2)$ is diffeomorphic to S^3. So we have

$$H^i(SU(2); \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

Letting x_3 to be a generator of $H^3(SU(2))$, we have $x_3^2 = 0$. So

$$H^*(SU(2); \mathbb{R}) \cong \Lambda[x_3] \quad \text{with degree}(x_3) = 3\,.$$

We proceed by induction on n. So we assume the statement to be true for $n = m$. So we have

$$H^*(SU(m); \mathbb{R}) \cong \Lambda[x_3, x_5, \ldots, x_{2m-1}] \qquad \text{with degree}(x_i) = i\,.$$

So by graded ring structure of $H^*(SU(m); \mathbb{R})$, we have $x_i \in H^i(SU(m); \mathbb{R})$. Now we prove the statement for $n = m + 1$. Recall by proposition (11.0.1) we have the Fiber bundle :

$$U(m) \hookrightarrow U(m+1) \xrightarrow{f} S^{2m+1}$$

with $f(A) = Ae_1$, where $\{e_1, \ldots, e_{m+1}\}$ is the standard ordered basis of $\mathbb{C}^{m+1}$ over $\mathbb{C}$. If we restrict f on $SU(m+1)$, it is easy to check that $f|_{SU(m+1)}$ is again proper submersion onto S^{2m+1}. So we get the locally trivial fiber bundle :

$$SU(m) \hookrightarrow SU(m+1) \xrightarrow{f} S^{2m+1}\,.$$

with $SU(m)$ connected and S^{2m+1} simply connected. So by theorem (11.0.4) we will get a Leray-Serre spectral sequnece of algebras $\{E_r^{*,*}, d_r\}$ where

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1} \quad \text{and} \quad E_{r+1}^{p,q} = \ker d_r^{p,q} / \operatorname{Im} d_r^{p-r,q+r-1}$$

and $E_2^{p,q} \cong H^p(S^{2m+1};\mathbb{R}) \otimes H^q(SU(m);\mathbb{R})$ and the spectral sequence converges to $H^*(SU(m+1);\mathbb{R})$ as an algebra.

By construction of spectral sequence we have

$$E_2^{*,*} \cong H^*(S^{2m+1};\mathbb{R}) \otimes H^*(SU(m);\mathbb{R})$$

isomorphic as graded algebras.

Let x_{2m+1} be a generator of $H^{2m+1}(S^{2m+1};\mathbb{R})$ so that $H^*(S^{2m+1};\mathbb{R}) \cong \Lambda[x_{2m+1}]$ isomorphic as graded algebra where degree$(x_{2m+1}) = 2m+1$. So

$$E_2^{*,*} \cong \Lambda[x_3, x_5, \dots, x_{2m-1}] \otimes \Lambda[x_{2m+1}] \cong \Lambda[x_3, x_5, \dots, x_{2m+1}]$$

isomorphic as graded algebras and degree of x_i is i.

Claim : For $r \geq 2$,

$$E_r^{*,*} \cong H^*(S^{2m+1};\mathbb{R}) \otimes H^*(SU(m);\mathbb{R}) \cong \Lambda[x_3, x_5, \dots, x_{2m+1}]$$

isomorphic as graded algebras and $E_r^{p,q} \cong H^p(S^{2m+1};\mathbb{R}) \otimes H^q(SU(m);\mathbb{R})$.

Proof : We have the result for $r = 2$. We proceed by induction on r. So we assume we have the result for $r = s \geq 2$ i.e.

$$E_s^{*,*} \cong H^*(S^{2m+1};\mathbb{R}) \otimes H^*(SU(m);\mathbb{R}) \cong \Lambda[x_3, x_5, \dots, x_{2m+1}]$$

isomorphic as graded algebras. We need to prove the result for $r = s+1$. We intend to prove that $d_s = 0$. Then it will follow that $E_{s+1}^{*,*} \cong E_s^{*,*}$ isomorphic as graded algebras. So the claim is proved by induction.

To show that $d_s = 0$: Since d_s is an antiderivation, it is enough to show that d_s takes the generators to 0 i.e. $d_s(x_i) = 0$ for $i = 3, 5, \dots, 2m+1$.

- For $i = 3, 5, \dots, 2m-1$,

$$x_i \in E_s^{0,i} \quad \text{and} \quad d_s^{0,i} : E_s^{0,i} \to E_s^{s,i-s+1}.$$

- ∘ If $s \neq 2m+1$, $H^s(S^{2m+1};\mathbb{R}) = 0$ and hence

$$E_s^{s,i-s+1} \cong H^s(S^{2m+1};\mathbb{R}) \otimes H^{i-s+1}(SU(m);\mathbb{R}) \cong 0$$

  So $d_s(x_i) = 0$

- ∘ If $s = 2m+1$, then $i - s + 1 = i - 2m < 0$. So $E_s^{s,i-s+1} \cong 0$ and hence $d_s(x_i) = 0$.

- For $i = 2m+1$,

$$x_{2m+1} \in E_s^{2m+1,0} \quad \text{and} \quad d_s^{2m+1,0} : E_s^{2m+1,0} \to E_s^{2m+1+s,1-s}.$$

  Now $1 - s < 0$, so $E_s^{2m+1+s,1-s} = 0$ and hence $d_s(x_{2m+1}) = 0$.

So we have $d_s(x_i) = 0$ for $i = 3, 5, \ldots, 2m+1$ and hence $d_s = 0$. So we have proved that for $r \geq 2$, $E_r^{*,*} \cong \Lambda[x_3, x_5, \ldots, x_{2m+1}]$ as graded algebras.

So

$$H^*(SU(m+1);\mathbb{R}) \cong E_\infty^{*,*} \cong \Lambda[x_3, x_5, \ldots, x_{2m+1}].$$

isomorphic as graded algebra with degree$(x_i) = i$.

So by induction we have proved that for $n > 1$,

$$H^*(SU(n);\mathbb{R}) \cong \Lambda[x_3, x_5, \ldots, x_{2n-1}]$$

isomorphic as graded algebras with degree$(x_i) = i$.

□