# Differential Topology - MA815

This file contains a list/short description of the material covered in each of the lectures.

- Lecture 1 - How to construct smooth manifolds by glueing open subsets of $\mathbb{R}^n$. Describe the smooth functions on these manifolds, smooth maps between smooth manifolds.

- Lecture 2 - Define tangent vector at a point $p \in M$ as derivations. Their description using coordinate functions. Given a smooth map $\varphi : X \to Y$ describe the push forward of tangent vectors, that is, the map $D\varphi|_p : T_pX \to T_{\varphi(p)}Y$. Construction of the tangent bundle. Chain rule implies "functoriality" for $D$.

- Lecture 3 - Example of the tangent bundle of $S^1$. Define vector bundles on a smooth manifold, the tangent bundle being an example of a vector bundle. More generally, the glueing construction can be used to construct fiber bundles. Define the dual bundle and the determinant bundle of a vector bundle. Cotangent vectors, their geometric significance in terms of differential forms. Description of cotangent vectors using coordinate functions. The cotangent bundle.

- Lecture 4 - The tangent bundle is an invariant of the smooth manifold. Define maps between vector bundles, sections of bundles, basic results about them. If $U$ is a coordinate open set then the tangent bundle over $U$ is the trivial bundle. Conversely, if the tangent bundle is trivial, does it mean that $U$ can be given global coordinates? Let $G$ be a group acting via smooth covering maps on a smooth manifold $X$. Then we give a manifold structure on $X/G$ in a "natural" way so that the map $X \to X/G$ becomes a smooth map. Use this in the case of $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ to see that the tangent bundle of $S^1$ is trivial, but it does not carry global coordinate functions ...

- Lecture 5 - The previous discussion continues ... Constructions on vector spaces. Direct sums, tensor products, tensor algbera, symmetric algebra, exterior algebra.

- Lecture 6 - Extend the above constructions to vector bundles. Sheaf of sections of a vector bundle. Glueing construction for sheaves.

- Lecture 7 - Defining the differential taking $i$ forms to $i + 1$ forms. Define $d$ using local coordinates and check that these glue together.

Basic properties of $d$. The de Rham complex, de Rham cohomology of a smooth manifold. Computing $H^0(X, \mathbb{R})$ when $X$ is connected.

- Lecture 8 - Computing $H^*(X, \mathbb{R})$ for $X = \mathbb{R}, S^1$. Ring structure de Rham cohomology. Begin to investigate properties of de Rham cohomology. Pull back of vector bundles and their properties.

- Lecture 9 - A smooth map $\varphi : X \to Y$ induces a map $\varphi^*$ of de Rham complexes and so gives rise to a map $H^*(Y, \mathbb{R}) \to H^*(X, \mathbb{R})$. Functoriality of pull back.

- Lecture 10 - Homotopic maps induce the same map on de Rham cohomology. Cohomology of $\mathbb{R}^n$ and $\mathbb{R}^2 \setminus (0, 0)$.

- Lecture 11 - Mayer-Vietoris sequence. Cohomology of $S^n$.

- Lecture 12 - Compactly supported sections of a vector bundle. Cohomology with compact supports. Computation of $H_c^*(\mathbb{R}, \mathbb{R})$. Mayer-Vietoris sequence for cohomology with compact support. Homotopy invariance for cohomology with compact support. Computation of $H_c^*(\mathbb{R}^n, \mathbb{R})$.

- Lecture 13 - Explicit isomorphism $H_c^n(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ using integration. Use this to define degree of a proper map $\mathbb{R}^n \to \mathbb{R}^n$. Show that degree is an integer. Define oriented manifolds, and integration on them.

- Lecture 14 - Define manifold with boundary, prove Stokes Theorem.

- Lecture 15 - Poincare duality (proof only for manifolds of finite type).

- Lecture 16 - Let $M$ be an oriented manifold and let $S \subset M$ be a compact and oriented embedded submanifold. Define a cohomology class associated to $S$. Compute class of a point and class of the manifold $M$. Kunneth formula.

- Lecture 17 - Homotopy property of vector bundles. Oriented vector bundles. Compact vertical cohomology $H_{cv}^*(E, \mathbb{R})$ and integration along fibers for an oriented vector bundle.

- Lecture 18 - Integration along fibers induces a map $H_{cv}^*(E, \mathbb{R}) \to H^{*-n}(M, \mathbb{R})$. Projection formula. Thom isomorphism for manifolds of finite type.

- Lecture 19 - Relate Poincare dual and Thom class. Apply this to show that $[\eta_S] = [\eta_{S_1}] \wedge [\eta_{S_2}]$, where $S = S_1 \cap S_2$ is a transverse intersection of two embedded submanifolds $S_1$ of $M$, and $M$, $S_1$, $S_2$ are oriented.

- Lecture 20 - The aim of this lecture is to see an interesting application of $[\eta_S] = [\eta_{S_1}] \wedge [\eta_{S_2}]$, a result proved in the previous lecture.

  The first part of this lectures describes how to give complex manifolds an orientation in a canonical way.

  The second part of the lecture shows that when $S_1$ and $S_2$ are two complex submanifolds of a complex manifold $M$ and they meet transversally, then the intersection $S = S_1 \cap S_2$ is a complex submanifold of $M$ in a canonical way.

  In Lecture 19 we gave an orientation to the intersection $S$. In view of the previous two parts of this lecture, $S$ has a canonical orientation as it is a complex manifold. Both these orientations coincide. This is left as a check to the reader.

  Now let $M$ be a compact complex manifold and let $S_1$ and $S_2$ be two closed complex submanifolds. Assume that they are in complementary dimensions and their intersection is transversal. Then we conclude, from the above discussions, that each point in the intersection receives a $+$ orientation. Using this we compute the ring structure of the cohomology of $\mathbb{P}^n_{\mathbb{C}}$ and show that the ring is generated by class of the hyperplane. Finally, this is used to prove Bezout's Theorem on intersection of smooth hypersurfaces in $\mathbb{P}^2_{\mathbb{C}}$.

- Lecture 21 - Define $c_1(L)$ for a complex line bundle $L$ on $X$. Use this, the previous lecture and Leray-Hirsch Theorem to compute cohomology $H^*(\mathbb{P}(E), \mathbb{R})$, where $E$ is a complex vector bundle on $X$. (This is not mentioned in the lecture, but we can now easily define $c_i(E) \in H^{2i}(X, \mathbb{R})$, the Chern classes of $E$.) Define Poincare polynomial of a manifold. Compute the Poincare polynomial of $H^*(\mathbb{P}(E), \mathbb{R})$. The main application is to compute cohomology of Grassmannians.