MA-207 Differential Equations II

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Some course policies

EVALUATION: 50 marks are waiting to be earned:

Quiz	20 marks
End Semester exam	30 marks
Total	**50** marks

The quiz will be held in the tutorial on 12th October, 2021.

FR grade if marks are $<$ **15/50**.

Any form of academic dishonesty will invite severe penalties.

Books

Elementary differential equations with boundary value problems
by William F. Trench (available online)

Differential Equations with Applications and Historical Notes
by George F. Simmons

Aim of this course

The aim of this course is to see some methods to find solutions to differential equations.

There are two parts in this course.

1. In the first part we shall see how to solve differential equations in one variable.
2. In the second part we shall see how to solve differential equations involving functions of two variables.

In both parts we shall find solutions to the differential equations as series.

In the first part, these series will usually be power series in one variable. In the second part, we will consider more complicated kinds of series, for example, Fourier series.

Aim of this course

A very beautiful, simple and powerful technique we will learn in this course is the Method of Separation of Variables. This will come towards the end of the course.

Separation of variables, combined with the series representation, yields a way to solve some PDE's, which otherwise will be incredibly hard to solve.

For example, try to solve the following PDE (heat equation) on your own

$$\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
&u(0,t) = 0 && t \geq 0 \\
&u(L,t) = 0, && t \geq 0 \\
&u(x,0) = x(L-x), && 0 \leq x \leq L
\end{aligned}$$

More generally, instead of $x(L-x)$ we could have taken any "nice" function. We will learn in the last few lectures how to solve this PDE. This ends a very brief introduction and we now begin the course with a study of power series.

Consider an ODE

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x), q(x)$ continuous.

Let $y_1(x)$ be one solution of the above ODE.

We can try to use the method of variation of parameters to find another linearly independent solution, that is, put

$$y_2(x) = u(x)y_1(x)$$

in the ODE and solve for $u(x)$.

Question. How to find one solution?

For this, we will solve our ODE in terms of power series.

Definition

For real numbers $x_0, a_0, a_1, a_2, \ldots$, an infinite series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n := a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots.$$

is called a power series in $x - x_0$ with center x_0.

For a real number x_1, if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x_1 - x_0)^n$$

exists and is finite, then we say the power series converges at the point $x = x_1$.
In this case, the value of the series at x_1 is, by definition, the value of the limit.

Power series

Definition

If the series does not converge at x_1, that is, either the limit does not exist, or it is $\pm\infty$, then we say the power series diverges at x_1.

Obviously, a power series always converges at its center $x = x_0$.

Power series - Radius of convergence

Theorem

For any power series,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

exactly one of these statements is true.

1. *The power series converges only for* $x = x_0$.
2. *The power series converges for all values of* x.
3. *There is a positive number* $0 < R < \infty$ *such that the power series converges if* $|x - x_0| < R$ *and diverges if* $|x - x_0| > R$.

R is called the radius of convergence of the power series.

Define $R = 0$ in case (i)
Define $R = \infty$ in case (ii).

Question. How to compute the radius of convergence?

There are two methods to do this.

Theorem

(Ratio test) Assume that there is an integer N such that for all $n \geq N$ we have $a_n \neq 0$. Also assume the following limit exists

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and denote it by L.

Then radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is

$\boxed{R = 1/L}$.

For $L = 0$, we get $R = \infty$
and for $L = \infty$, we get $R = 0$.

The ratio test will not work for all series (for example, when many of the a_n's are 0).

However, the root test, which is the second method to compute the radius of convergence, will work for all power series. We first need to recall the definition of limsup.

Definition

Suppose we are given a sequence $\{a_n\}_{n\geq 1}$.
For every $k \geq 1$ define

$$b_k := \sup_{n\geq k}\{a_n\}\,.$$

Convince yourself that $\{b_k\}_{k\geq 1}$ is a decreasing sequence, that is,

$$b_1 \geq b_2 \geq b_3 \geq \dots$$

Define

$$\limsup\{a_n\} := \lim_{n\to\infty} b_n\,.$$

Definition

Similarly, define $\liminf\{a_n\}$, by replacing $\sup$ by $\inf$ in the above discussion.

Remark

Note that for a sequence $\{a_n\}_{n\geq 1}$, the limit may not exist. However, the $\limsup$ and $\liminf$ always exist (possibly $+\infty$ or $-\infty$).

Theorem

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if $\limsup\{a_n\} = \liminf\{a_n\}$.
Further, if $\lim_{n\to\infty} a_n$ exists, then

$$\limsup\{a_n\} = \liminf\{a_n\} = \lim_{n\to\infty} a_n$$

Strictly speaking, when we say that $\lim_{n\to\infty} a_n$ exists, we mean that this limit exists and is finite.

However, sometimes we shall be a little careless and say that $\lim_{n\to\infty} a_n$ exists in the following cases also: if $\lim_{n\to\infty} a_n = \infty$ or $\lim_{n\to\infty} a_n = -\infty$.

Recall, for example, the definition of $\lim_{n\to\infty} a_n = \infty$. For every $N \in \mathbb{R}$, there exists $n(N) \geq 1$ (that is, n depends on N) such that $a_k \geq N$ for all $k \geq n(N)$.

For example, convince yourself that for the sequence defined by $b_{2n-1} := n$ and $b_{2n} := n - 1$ $(n \geq 1)$, we have $\lim_{n\to\infty} b_n = \infty$

Now that we have recalled the definition of limsup, we return to the root test.

Theorem

(Root test)Let $\limsup\{|a_n|^{1/n}\} = L$*.*

Then radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ *is*

$\boxed{R = 1/L}$.

For $L = 0$*, we get* $R = \infty$*.*
For $L = \infty$*, we get* $R = 0$*.*

This concludes the discussion on how to compute the radius of convergence of a power series.

Power series - Radius of convergence

Theorem

Let $R > 0$ be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$.

For $R = \infty$, we write $(x_0 - R, x_0 + R) = (-\infty, \infty) = \mathbb{R}$.

Definition

The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.

Example

Find the radius of convergence and interval of convergence (if $R > 0$) of

$$\sum_{0}^{\infty} n! x^n .$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} (n+1) = \infty .$$

So $R = 1/\infty = 0$.

Power series - examples

Example

Find the radius of convergence and interval of convergence (if $R > 0$) of

$$\sum_{10}^{\infty} (-1)^n \frac{x^n}{n^n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| = 0$$

So $R = 1/0 = \infty$. Interval of convergence $(-\infty, \infty)$.

Power series - examples

Example

Find the radius of convergence and interval of convergence (if $R > 0$) of

$$\sum_{0}^{\infty} 2^n n^3 (x-1)^n$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)^3}{2^n n^3} \right| = 2$$

So $R = 1/2$. Interval of convergence $(1/2, 3/2)$.

Power series as functions

Theorem

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. We assume $\boxed{R > 0}$.

We define a function $f : (x_0 - R, x_0 + R) \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

This function satisfies the following properties

1. *f is infinitely differentiable $\forall\ x \in (x_0 - R, x_0 + R)$.*

Power series as functions

Theorem (continued ...)

2. *The successive derivatives of f can be computed by differentiating the power series term-by-term, that is,*

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} a_n (x - x_0)^n$$
$$= \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x - x_0)^n$$

Exchanging a differential operator and a sum/integral is something which needs to be done with care

$$= \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$

Power series as functions

Theorem (continued ...)

3. $$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\dots(n-k+1)\, a_n (x-x_0)^{n-k}$$
4. *The power series representing the derivatives $f^{(n)}(x)$ have same radius of convergence R.*
5. *We can determine the coefficients a_n (in terms of derivatives of f at x_0) as*
$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Power series as functions

Theorem (continued ...)

- *We can also integrate the function* $f(x) = \sum_0^\infty a_n (x - x_0)^n$

term-wise, that is,if $[a, b] \subset (x_0 - R, x_0 + R)$, *then*

$$\int_a^b f(x)\, dx = \sum_{n=0}^\infty a_n \int_a^b (x - x_0)^n\, dx = \sum_0^\infty \frac{a_n}{n+1} (x - x_0)^{n+1}$$

Exchanging an integral operator and a sum is something which needs to be done with care

Theorem

(i) Power series representation of f in an open interval I containing x_0 is unique, that is, if

$$f(x) = \sum_{0}^{\infty} a_n (x - x_0)^n = \sum_{0}^{\infty} b_n (x - x_0)^n$$

for all $x \in I$, then $a_n = b_n \ \forall \ n$.
(ii) If

$$\sum_{0}^{\infty} a_n (x - x_0)^n = 0$$

for all $x \in I$, then $a_n = 0$ for all n.

Proof. (i)

$$a_n = \frac{f^{(n)}(x_0)}{n!} = b_n \qquad \text{for all} \quad n.$$

It is clear that (ii) follows from (i).

Examples of power series

Example (Power series representation of some familiar functions)

(i) $$e^x = \sum_0^\infty \frac{x^n}{n!} \qquad -\infty < x < \infty$$

(ii) $$\sin x = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad -\infty < x < \infty$$

(iii) $$\frac{1}{1-x} = \sum_0^\infty x^n \qquad -1 < x < 1$$

(iv) $$\frac{d}{dx}(\sin x) = \sum_0^\infty (-1)^n \frac{d}{dx}\left(\frac{x^{2n+1}}{(2n+1)!}\right)$$

$$= \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

Algebraic operations on power series

Definition

If $\quad f(x) = \sum_0^{\infty} a_n (x - x_0)^n \qquad g(x) = \sum_0^{\infty} b_n (x - x_0)^n$

have radius of convergence R_1 and R_2, respectively, then

$$c_1 f(x) + c_2 g(x) := \sum_0^{\infty} (c_1 a_n + c_2 b_n)(x - x_0)^n$$

has radius of convergence $\boxed{R \geq \text{min } \{R_1, R_2\}}$ for $c_1, c_2 \in \mathbb{R}$.

Further, we can multiply the series as if they were polynomials, that is

$$f(x)g(x) = \sum_0^{\infty} c_n (x - x_0)^n; \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

It also has radius of convergence $R \geq \text{min } \{R_1, R_2\}$.

Algebraic operations on power series

Example

Find the power series expansion for $\cosh x$ in terms of powers of x^n.

$$\begin{aligned}\cosh x &= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{1}{2}\left[1+(-1)^n\right]\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\end{aligned}$$

Since radius of convergence for Taylor series of e^x and e^{-x} are ∞, the power series expansion of $\cosh x$ is valid on $\mathbb{R}$.

If $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ then $f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$.

Put $r = n - 1$ into $f'(x)$, we get

$$f'(x) = \sum_{r=0}^{\infty} (r+1) a_{r+1} (x - x_0)^r$$

Similarly,

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n (x - x_0)^{n-k} \\ &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\dots(n+1) a_{n+k} (x - x_0)^n \end{aligned}$$

Algebraic operations on power series

Example

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Write $(x-1)f''$ as a power series around 0.

$$
\begin{aligned}
(x-1)f'' &= xf'' - f'' \\
&= x\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n \\
&= \sum_{n=0}^{\infty} \left[(n+1)n a_{n+1} - (n+2)(n+1)a_{n+2}\right] x^n
\end{aligned}
$$

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Using power series to find formal solution to ODE's

Example

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

for all x in an open interval I containing $x_0 = 1$.

- Find the power series of y' and y'' in terms of $x-1$ in the interval I. Use these to express the function

  $$(1+x)y'' + 2(x-1)^2 y' + 3y$$

  as a power series in $x-1$ on I.
- Find necessary and sufficient conditions on the coefficients a_n's, so that $y(x)$ is a formal solution of the ODE

  $$(1+x)y'' + 2(x-1)^2 y' + 3y = 0$$

Example (continued ...)

Solution. Write the ODE in $(x-1)$, that is

$$(1+x)y'' + 2(x-1)^2 y' + 3y = (x-1)y'' + 2y'' + 2(x-1)^2 y' + 3y$$

Express each of $(x-1)y''$, $2y''$, $2(x-1)^2 y'$ and $3y$ as a power series in powers of $(x-1)$ and add them.

$$\begin{aligned}(x-1)y'' &= (x-1)\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} \\ &= \sum_{n=1}^{\infty} (n+1)na_{n+1}(x-1)^n \\ &= \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n\end{aligned}$$

Using power series to find formal solution to ODE's

Example (continued ...)

$$2y'' = \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2}$$

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n$$

$$2(x-1)^2 y' = 2(x-1)^2 \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$

$$= \sum_{n=1}^{\infty} 2na_n(x-1)^{n+1}$$

$$= \sum_{n=2}^{\infty} 2(n-1)a_{n-1}(x-1)^n$$

$$= \sum_{n=0}^{\infty} 2(n-1)a_{n-1}(x-1)^n \quad (a_{-1} = 0)$$

Example (continued ...)

We have

$$(x-1)y'' = \sum_{n=0}^{\infty}(n+1)na_{n+1}(x-1)^n$$

$$2y'' = \sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}(x-1)^n$$

$$2(x-1)^2y' = \sum_{n=0}^{\infty}2(n-1)a_{n-1}(x-1)^n \quad (a_{-1}=0)$$

Now we get

$$(x-1)y'' + 2y'' + 2(x-1)^2y' + 3y = \sum_{n=0}^{\infty}b_n(x-1)^n$$

where

$$b_n = (n+1)na_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n$$

Example (continued ...)

For the second part,

$$y(x) = \sum_{0}^{\infty} a_n (x-1)^n \text{ is the solution of the ODE}$$

$$(x-1)y'' + 2y'' + 2(x-1)^2 y' + 3y = 0$$

on the open interval I containing 1 if and only if

$$\sum_{n=0}^{\infty} b_n (x-1)^n = 0 \text{ on } I \iff b_n = 0 \quad \text{for all } n$$

that is, a_n's satisfy the following recursive relation

$$(n+1)na_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n = 0$$

for all n.

Example (continued ...)

Thus, we can take any coefficients which satisfy the above relations, and we get a power series which satisfies the above equation. However, note that this power series may not define a nice function. In this sense, the above solution is "formal".

Taylor series

Definition

Let $f(x)$ be an infinitely differentiable at x_0. The Taylor series of f at x_0 is defined as the power series

$$TS\, f|_{x_0} := \sum_0^\infty \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Let us make an observation.

Suppose f is defined by a power series in an interval of x_0, that is, $f(x) = \sum_{n \geq 0} a_n (x - x_0)^n$ in the interval $(x_0 - R, x_0 + R)$. When we apply the above definition of Taylor series, we see that

$$TSf|_{x_0} = \sum_0^\infty a_n (x - x_0)^n = f(x)\,.$$

Thus, in this case from the Taylor series we get back the function f.

However, the class of infinitely differentiable functions is larger than the class of power series and the above need not be true for infinitely differentiable functions in an interval around x_0

Example

The function $$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is infinitely differentiable at 0. But $f^{(n)}(0) = 0$ for all n.

Hence the Taylor series of f at 0 is the constant function taking value 0.

Therefore Taylor series of f at 0 does not converge to function $f(x)$ on any open interval around 0.

Definition

Suppose

- $f(x)$ is infinitely differentiable at x_0; and
- Taylor series of f at x_0 converges to $f(x)$ for all x in some open interval around x_0;

Then f is called analytic at x_0.

Thus, if f is analytic, then there is an interval I around x_0 and f is given by a power series in I.

Example

The function $$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is not analytic at 0. Here 2nd condition fails. However, f is analytic at all $x \neq 0$.

Analytic functions

Example

1. Polynomials, e^x, $\sin x$ and $\cos x$ are analytic at all $x \in \mathbb{R}$.
2. $f(x) = \tan x$ is analytic at all x except $x = (2n+1)\pi/2$, where $n = \pm 1, \pm 2, \ldots$.
3. $f(x) = x^{5/3}$ is analytic at all x except $x = 0$.

Theorem (Analytic functions)

1. *If* $f(x)$ *and* $g(x)$ *are analytic at* x_0*, then*
   $f(x) \pm g(x) \qquad f(x)g(x) \qquad f(x)/g(x) \quad$ *(if* $g(x_0) \neq 0$*)*
   are analytic at x_0*.*
2. *If* $f(x)$ *is analytic at* x_0 *and* $g(x)$ *is analytic at* $f(x_0)$*, then* $g(f(x)) := (g \circ f)(x)$ *is analytic at* x_0*.*
3. *If a power series* $\sum_0^\infty a_n(x - x_0)^n$ *has radius of convergence* $R > 0$*, then the function* $f(x) := \sum_0^\infty a_n(x - x_0)^n$ *is analytic at all points* $x \in (x_0 - R, x_0 + R)$*.*

Analytic functions

Example

The function $f(x) = x^2 + 1$ is analytic everywhere. Since $x^2 + 1$ is never 0, the function $h(x) := \frac{1}{x^2+1}$ is analytic everywhere. However, there is no power series around 0 which represents $h(x)$ on $\mathbb{R}$.

If there were such a power series, then by uniqueness, it has to be the power series expansion of $h(x)$ around 0, which is

$$1 - x^2 + x^4 - x^6 + \cdots$$

However, the radius of convergence of this is $R = 1$.

In fact, for any x_0, there is no power series around x_0 which represents $h(x)$ everywhere.

Analytic functions

Theorem

Let

$$F(x) = \frac{N(x)}{D(x)} \quad \textit{example} \quad F(x) = \frac{x^3 - 1}{x^2 + 1}$$

be a rational function, where $N(x)$ and $D(x)$ are polynomials without any common factors, that is they do not have any common (complex) zeros. Let $\alpha_1, \ldots, \alpha_r$ be distinct complex zeros of $D(x)$.

Then $F(x)$ is analytic at all x except at $x \in \{\alpha_1, \ldots, \alpha_r\}$.If x_0 is different from $\{\alpha_1, \ldots, \alpha_r\}$, then the radius of convergence R of the Taylor series of F at x_0

$$TS\, F_{x_0} = \sum_{0}^{\infty} \frac{F^{(n)}(x_0)}{n!}(x - x_0)^n$$

Analytic functions

Theorem (continued . . .)

is given by

$$R = \mathit{min}\,\{|x_0 - \alpha_1|, |x_0 - \alpha_2|, \ldots, |x_0 - \alpha_r|\}$$

Example

If

$$F(x) = \frac{N(x)}{D(x)} = \frac{(2+3x)}{(4+x)(9+x^2)}$$

then $D(x)$ has zeros at -4 and $\pm 3\iota$, where $\iota = \sqrt{-1}$.

Hence F is analytic at all x except at $x \in \{-4, \pm 3\iota\}$.

If $x = 2$, then radius of convergence of Taylor series of F at $x = 2$ is

$$\min\{|2+4|, |2+3\iota|, |2-3\iota|\} = \min\{6, \sqrt{13}\} = \sqrt{13}$$

If $x = -6$, then radius of convergence of Taylor series of F at $x = -6$ is

$$\min\{|-6+4|, |-6 \pm 3\iota|\} = \min\{2, \sqrt{45}\} = 2$$

Power series solution of ODE

Theorem (Existence Theorem)

If $p(x)$ and $q(x)$ are analytic functions at x_0, then every solution of

$$y'' + p(x)y' + q(x)y = 0$$

is also analytic at x_0; and therefore any solution can be expressed as

$$y(x) = \sum_{0}^{\infty} a_n (x - x_0)^n$$

If R_1 = radius of convergence of Taylor series of $p(x)$ at x_0,
R_2 = radius of convergence of Taylor series of $q(x)$ at x_0,
then radius of convergence of $y(x)$ is at least min$(R_1, R_2) > 0$.

In most applications, $p(x)$ and $q(x)$ are rational functions, that is quotient of polynomial functions.

Example

Let us solve $\boxed{y'' + y = 0 \qquad (1)}$ by power series method.
Compare with $y'' + p(x)y' + q(x)y = 0$,
$p(x) = 0$ and $q(x) = 1$ are analytic at all x.
We can find power series solution in $x - x_0$ for any x_0.
Let us assume $x_0 = 0$ for simplicity.
By existence theorem, all solution of (1) can be found in the form

$$y(x) = \sum_{0}^{\infty} a_n x^n$$

and the series will have ∞ radius of convergence.
Since

$$y'' = \sum_{2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Example (Continue ...)

$$y'' + y = \sum_{0}^{\infty}((n+2)(n+1)a_{n+2} + a_n)x^n = 0$$

By uniqueness of power series in $x - x_0$ we get the recursion formula

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$\implies a_{n+2} = \frac{-1}{(n+2)(n+1)}a_n \quad \forall n$$

Therefore,

$$a_2 = \frac{-1}{2.1}a_0, \quad a_4 = \frac{-1}{4.3}a_2 = \frac{1}{4!}a_0 \ \ldots \ \ a_{2n} = (-1)^n \frac{1}{(2n)!}a_0$$

$$a_3 = \frac{-1}{3.2}a_1, \quad a_5 = \frac{-1}{5.4}a_3 = \frac{1}{5!}a_1 \ \ldots \ \ a_{2n+1} = (-1)^n \frac{1}{(2n+1)!}a_1$$

Example (Continue ...)

Define

$$y_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \qquad (a_0 = 1, a_1 = 0)$$

$$y_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \qquad (a_0 = 0, a_1 = 1)$$

Then

$$y(x) = \sum_{0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

is a general solution of the ODE (1).

In this case, $y_1(x) = \cos x$ and $y_2(x) = \sin x$.
We don't need to check the series for converges, since the existence theorem guarantees that the series converges for all x.

Power series solution of ODE

In this course, we will consider ODE

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

with $P_i(x)$ polynomials for $i = 0, 1, 2$ without any common factor. If we write ODE in the standard form

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and if x_0 is not a zero of $P_0(x)$, then $P_1(x)/P_0(x)$ and $P_2(x)/P_0(x)$ will be analytic at x_0, hence, we can find the series solution of ODE in the form

$$y(x) = \sum_{0}^{\infty} a_n(x - x_0)^n$$

Steps for Series solution of linear ODE

1. Write ODE in standard form $y'' + p(x)y' + q(x)y = 0$.
2. Choose x_0 at which $p(x)$ and $q(x)$ are analytic. If boundary conditions at x_0 are given, choose the center of the power series as x_0.
3. Find minimum of radius of convergence of Taylor series of $p(x)$ and $q(x)$ at x_0.
4. Let $y(x) = \sum_{0}^{\infty} a_n (x - x_0)^n$, compute the power series for $y'(x)$ and $y''(x)$ at x_0 and substitute these into the ODE.
5. Set the coefficients of $(x - x_0)^n$ to zero and find recursion formula.
6. From the recursion formula, obtain (linearly independent) solutions $y_1(x)$ and $y_2(x)$. The general solution then looks like $y(x) = a_1 y_1(x) + a_2 y_2(x)$.

Example

Find the power series in x for the general solution of

$$(1+2x^2)y''+6xy'+2y=0$$

Solution. Note that 0 is not a zero of $P_0(x)=1+2x^2$, hence, the series solution in powers of x exists.

Putting $y=\sum_0^\infty a_n x^n$ in the ODE, we get

$$\begin{aligned}
&(1+2x^2)y''+6xy'+2y\\
&=y''+2x^2y''+6xy'+2y\\
&=\sum_0^\infty((n+2)(n+1)a_{n+2}+2n(n-1)a_n+6na_n+2a_n)x^n
\end{aligned}$$

$$\implies (n+2)(n+1)a_{n+2}+[2n(n-1)+6n+2]a_n=0$$

Example (Continue ...)

$$\implies a_{n+2} = -\frac{2n^2+4n+2}{(n+2)(n+1)}\, a_n = -2\frac{n+1}{(n+2)}\, a_n \quad n \geq 0$$

Since indices on left and right differ by 2, we write separately for $n = 2m$ and $n = 2m+1$, $m \geq 0$, so

$$a_{2m+2} = -2\frac{2m+1}{2m+2}\, a_{2m} = -\frac{2m+1}{m+1}\, a_{2m}$$

$$a_{2m+3} = -2\frac{2m+2}{2m+3}\, a_{2m+1} = -4\frac{m+1}{2m+3}\, a_{2m+1}$$

$$a_2 = -\frac{1}{1}\, a_0$$

$$a_4 = -\frac{3}{2}\, a_2 = \frac{1.3}{1.2}\, a_0$$

$$a_6 = -\frac{5}{3}\, a_4 = -\frac{1.3.5}{1.2.3}\, a_0$$

Example (Continue ...)

$$a_{2m} = (-1)^m \frac{1.3.5.\ldots.(2m-1)}{m!} a_0$$

$$= (-1)^m \frac{\prod_{j=1}^{m}(2j-1))}{m!} a_0$$

$$a_{2m+3} = -4\frac{m+1}{2m+3} a_{2m+1}$$

$$a_3 = -4\frac{1}{3} a_1$$

$$a_5 = -4\frac{2}{5} a_3 = 4^2\frac{1.2}{3.5} a_1$$

$$a_7 = -4\frac{3}{7} a_5 = -4^3\frac{1.2.3}{3.5.7} a_1$$

$$a_{2m+1} = (-1)^m 4^m \frac{m!}{\prod_{j=1}^{m}(2j+1)} a_1$$

Example (continued ...)

We can write the solution

$$y = \sum_{0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary scalars and

$$y_1(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!} x^{2m}$$

$$y_2(x) = \sum_{m=0}^{\infty} (-1) \frac{4^m m!}{\prod_{j=1}^{m}(2j+1)} x^{2m+1}$$

Since $P_0(x) = 1 + 2x^2$ has complex zeros $\frac{\pm\iota}{\sqrt{2}}$, the power series solution converges in the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. □

Example

Find the coefficients $a_0, \ldots, a_6$ in the series solution

$$y = \sum_{0}^{\infty} a_n x^n$$

of the Initial Value Problem (IVP)

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0$$

with

$$y(0) = -1, \; y'(0) = -2.$$

Zeros of $P_0(x) = 1 + x + 2x^2$ are $\frac{1}{4}(-1 \pm \imath\sqrt{7})$ whose absolute values are $1/\sqrt{2}$. Hence the series solution to the IVP converges on the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Example (continued ...)

$$(1+x+2x^2)y''+(1+7x)y'+2y=\sum_0^\infty b_n x^n=0$$

$$\begin{aligned} b_n &= (n+2)(n+1)a_{n+2}+(n+1)na_{n+1}+2n(n-1)a_n \\ &\quad +(n+1)a_{n+1}+7na_n+2a_n=0 \end{aligned}$$

that is

$$(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}+(2n^2+5n+2)a_n=0$$

Since $2n^2+5n+2=(n+2)(2n+1)$,

$$a_{n+2}=-\frac{n+1}{n+2}\,a_{n+1}-\frac{2n+1}{n+1}\,a_n \qquad n\geq 0$$

Example (continued ...)

$$a_{n+2} = -\frac{n+1}{n+2}\,a_{n+1} - \frac{2n+1}{n+1}\,a_n \qquad n \geq 0$$

From the initial conditions $y(0) = -1,\; y'(0) = -2$ we get

$$a_0 = y(0) = -1, \quad a_1 = y'(0) = -2$$

$$a_2 = -\frac{1}{2}a_1 - a_0 = 2$$

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = \frac{5}{3}$$

Check that

$$y(x) = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 + \dots$$

Slightly more complicated ODE's

The best possible situation is when we have an ODE of the form

$$y'' + p(x)y' + q(x) = 0$$

where $p(x)$ and $q(x)$ are analytic in an interval around x_0.

We just saw how to solve such ODE's.

We will next consider in detail the Legendre equation (an ODE which falls in the above category of "nice" ODE's)

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0\,.$$

However, there are other ODE's which occur naturally, which do not fall into the above "nice" category, and which we would like to solve. For example, Bessel's equation :

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

We will see later how to solve some such ODE's.