MA-207 Differential Equations II

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Powai, Mumbai - 76

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Some course policies

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EVALUATION: 50 marks are waiting to be earned:

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Any form of academic dishonesty will invite severe penalties.

Books

Elementary differential equations with boundary value problems
by William F. Trench (available online)

Differential Equations with Applications and Historical Notes
by George F. Simmons

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In both parts we shall find solutions to the differential equations as series.

In the first part, these series will usually be power series in one variable. In the second part, we will consider more complicated kinds of series, for example, Fourier series.

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\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
&u(0,t) = 0 && t \geq 0 \\
&u(L,t) = 0, && t \geq 0 \\
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More generally, instead of $x(L-x)$ we could have taken any "nice" function. We will learn in the last few lectures how to solve this PDE.

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More generally, instead of $x(L-x)$ we could have taken any "nice" function. We will learn in the last few lectures how to solve this PDE. This ends a very brief introduction and we now begin the course with a study of power series.

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with $p(x), q(x)$ continuous.

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For this, we will solve our ODE in terms of power series.

Definition

For real numbers $x_0, a_0, a_1, a_2, \ldots$, an infinite series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n := a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots .$$

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For a real number x_1, if the limit

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (x_1 - x_0)^n$$

exists and is finite, then we say the power series converges at the point $x = x_1$.

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In this case, the value of the series at x_1 is, by definition, the value of the limit.

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Obviously, a power series always converges at its center $x = x_0$.

Power series - Radius of convergence

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1. *The power series converges only for* $x = x_0$.
2. *The power series converges for all values of* x.
3. *There is a positive number* $0 < R < \infty$ *such that the power series converges if* $|x - x_0| < R$ *and diverges if* $|x - x_0| > R$.

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Define $R = 0$ in case (i)
Define $R = \infty$ in case (ii).

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$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and denote it by L.

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Note that for a sequence $\{a_n\}_{n\geq 1}$, the limit may not exist. However, the $\limsup$ *and* $\liminf$ *always exist (possibly* $+\infty$ *or* $-\infty$*).*

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Theorem

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if $\limsup\{a_n\} = \liminf\{a_n\}$.
Further, if $\lim_{n\to\infty} a_n$ exists, then

$$\limsup\{a_n\} = \liminf\{a_n\} = \lim_{n\to\infty} a_n$$

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Recall, for example, the definition of $\lim_{n\to\infty} a_n = \infty$. For every $N \in \mathbb{R}$, there exists $n(N) \geq 1$ (that is, n depends on N) such that $a_k \geq N$ for all $k \geq n(N)$.

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For example, convince yourself that for the sequence defined by $b_{2n-1} := n$ and $b_{2n} := n - 1$ $(n \geq 1)$, we have $\lim_{n\to\infty} b_n = \infty$

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This concludes the discussion on how to compute the radius of convergence of a power series.

Power series - Radius of convergence

Theorem

Let $R > 0$ be the radius of convergence of the power series

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Then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$.

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Definition

The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.

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Power series - examples

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So $R = 1/\infty = 0$.

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So $R = 1/2$.

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$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

This function satisfies the following properties

1. *f is infinitely differentiable $\forall\ x \in (x_0-R, x_0+R)$.*

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Theorem (continued ...)

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Exchanging a differential operator and a sum/integral is something which needs to be done with care

$$= \sum_{n=0}^{\infty} n a_n(x-x_0)^{n-1}$$

Power series as functions

Theorem (continued ...)

3. $$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\ldots(n-k+1)\, a_n (x-x_0)^{n-k}$$

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$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Power series as functions

Theorem (continued ...)

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$$\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} a_n \int_a^b (x - x_0)^n\,dx = \sum_{0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}$$

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It is clear that (ii) follows from (i).

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$$c_1 f(x) + c_2 g(x) := \sum_{0}^{\infty} (c_1 a_n + c_2 b_n)(x - x_0)^n$$

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Find the power series expansion for $\cosh x$ in terms of powers of x^n.

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Since radius of convergence for Taylor series of e^x and e^{-x} are ∞, the power series expansion of $\cosh x$ is valid on $\mathbb{R}$.

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Similarly,

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n (x - x_0)^{n-k} \\ &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\dots(n+1) a_{n+k} (x - x_0)^n \end{aligned}$$

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&= \sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n \\
&= \sum_{n=0}^{\infty} \left[(n+1)n a_{n+1} - (n+2)(n+1)a_{n+2}\right] x^n
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Using power series to find formal solution to ODE's

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Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

for all x in an open interval I containing $x_0 = 1$.

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- Find the power series of y' and y'' in terms of $x-1$ in the interval I. Use these to express the function

$$(1+x)y'' + 2(x-1)^2 y' + 3y$$

as a power series in $x-1$ on I.

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- Find necessary and sufficient conditions on the coefficients a_n's, so that $y(x)$ is a formal solution of the ODE

  $$(1+x)y'' + 2(x-1)^2 y' + 3y = 0$$

Example (continued ...)

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Example (continued . . .)

$$2y'' = \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2}$$

Example (continued ...)

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Using power series to find formal solution to ODE's

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Now we get

$$(x-1)y'' + 2y'' + 2(x-1)^2y' + 3y = \sum_{n=0}^{\infty}b_n(x-1)^n$$

where

$$b_n = (n+1)na_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n$$

Example (continued ...)

For the second part,

$$y(x) = \sum_{0}^{\infty} a_n (x-1)^n \text{ is the solution of the ODE}$$

$$(x-1)y'' + 2y'' + 2(x-1)^2 y' + 3y = 0$$

on the open interval I containing 1 if and only if

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that is, a_n's satisfy the following recursive relation

$$(n+1)na_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n = 0$$

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Example (continued . . .)

Thus, we can take any coefficients which satisfy the above relations, and we get a power series which satisfies the above equation.

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Thus, we can take any coefficients which satisfy the above relations, and we get a power series which satisfies the above equation. However, note that this power series may not define a nice function. In this sense, the above solution is "formal".

Taylor series

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Let $f(x)$ be an infinitely differentiable at x_0.

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Let us make an observation.

Suppose f is defined by a power series in an interval of x_0, that is, $f(x) = \sum_{n\geq 0} a_n(x - x_0)^n$ in the interval $(x_0 - R, x_0 + R)$. When we apply the above definition of Taylor series, we see that

$$TSf|_{x_0} = \sum_{0}^{\infty} a_n(x - x_0)^n = f(x)\,.$$

Thus, in this case from the Taylor series we get back the function f.

Taylor series

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The function $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

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Therefore Taylor series of f at 0 does not converge to function $f(x)$ on any open interval around 0.

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The function $$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is not analytic at 0. Here 2nd condition fails. However, f is analytic at all $x \neq 0$.

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Analytic functions

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2. $f(x) = \tan x$ is analytic at all x except $x = (2n+1)\pi/2$, where $n = \pm 1, \pm 2, \ldots$.
3. $f(x) = x^{5/3}$ is analytic at all x except $x = 0$.

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3. *If a power series* $\displaystyle\sum_0^\infty a_n(x-x_0)^n$ *has radius of convergence* $R > 0$*, then the function* $f(x) := \displaystyle\sum_0^\infty a_n(x-x_0)^n$ *is analytic at all points* $x \in (x_0 - R, x_0 + R)$*.*

Analytic functions

Example

The function $f(x) = x^2 + 1$ is analytic everywhere. Since $x^2 + 1$ is never 0, the function $h(x) := \frac{1}{x^2+1}$ is analytic everywhere.
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If there were such a power series, then by uniqueness, it has to be the power series expansion of $h(x)$ around 0, which is

$$1 - x^2 + x^4 - x^6 + \cdots$$

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In fact, for any x_0, there is no power series around x_0 which represents $h(x)$ everywhere.

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Let

$$F(x) = \frac{N(x)}{D(x)}$$

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Then $F(x)$ is analytic at all x except at $x \in \{\alpha_1, \dots, \alpha_r\}$.If x_0 is different from $\{\alpha_1, \dots, \alpha_r\}$, then the radius of convergence R of the Taylor series of F at x_0

$$TS\, F_{x_0} = \sum_{0}^{\infty} \frac{F^{(n)}(x_0)}{n!}(x - x_0)^n$$

Analytic functions

Theorem (continued . . .)

is given by

$$R = \mathit{min}\,\{|x_0 - \alpha_1|, |x_0 - \alpha_2|, \ldots, |x_0 - \alpha_r|\}$$

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If $x = 2$, then radius of convergence of Taylor series of F at $x = 2$ is

$$\min\{|2+4|, |2+3\iota|, |2-3\iota|\} = \min\{6, \sqrt{13}\} = \sqrt{13}$$

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If $x = -6$, then radius of convergence of Taylor series of F at $x = -6$ is

$$\min\{|-6+4|, |-6 \pm 3\iota|\} = \min\{2, \sqrt{45}\} = 2$$

Theorem (Existence Theorem)

Power series solution of ODE

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If $p(x)$ and $q(x)$ are analytic functions at x_0, then every solution of

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If $R_1 =$ radius of convergence of Taylor series of $p(x)$ at x_0,
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then radius of convergence of $y(x)$ is at least min$(R_1, R_2) > 0$.

In most applications, $p(x)$ and $q(x)$ are rational functions, that is quotient of polynomial functions.

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Since

$$y'' = \sum_{2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

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Therefore,

$$a_2 = \frac{-1}{2.1}a_0, \quad a_4 = \frac{-1}{4.3}a_2 = \frac{1}{4!}a_0 \quad \ldots \quad a_{2n} = (-1)^n \frac{1}{(2n)!}a_0$$

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$$y_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \qquad (a_0 = 1, a_1 = 0)$$

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We don't need to check the series for converges, since the existence theorem guarantees that the series converges for all x.

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$$y(x) = \sum_0^\infty a_n(x - x_0)^n$$

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6. From the recursion formula, obtain (linearly independent) solutions $y_1(x)$ and $y_2(x)$. The general solution then looks like $y(x) = a_1y_1(x) + a_2y_2(x)$.

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$$\implies (n+2)(n+1)a_{n+2}+[2n(n-1)+6n+2]a_n=0$$

Example (Continue ...)

$$\implies a_{n+2} = -\frac{2n^2 + 4n + 2}{(n+2)(n+1)}\, a_n = -2\frac{n+1}{(n+2)}\, a_n \quad n \geq 0$$

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Example (Continue ...)

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Example (continued ...)

We can write the solution

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Since $P_0(x) = 1 + 2x^2$ has complex zeros $\frac{\pm\iota}{\sqrt{2}}$, the power series solution converges in the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. □

Example

Find the coefficients $a_0, \ldots, a_6$ in the series solution

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of the Initial Value Problem (IVP)

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0$$

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$$y(0) = -1, \; y'(0) = -2.$$

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Check that

$$y(x) = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 + \dots$$

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However, there are other ODE's which occur naturally, which do not fall into the above "nice" category, and which we would like to solve. For example, Bessel's equation :

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

We will see later how to solve some such ODE's.