MA-207 Differential Equations II

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September 7, 2021

Recap

The important things we did so far are

1. How to compute the radius of convergence of a power series
2. Power series defines a nice function in its interval of convergence
3. Suppose we are given an ODE: $y'' + p(x)y' + q(x)y = 0$, and $p(x)$ and $q(x)$ are analytic (given by power series) in an interval I around x_0, then the solution y is also analytic on I.
4. We can compute the two independent solutions, to an ODE as above, by plugging in a power series into the ODE and getting recursive relation for coefficients.

Legendre equation

The following ODE is known as the Legendre equation.

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

Here p denotes a fixed real number.

By Existence theorem, power series solution in x exists on the interval $(-1, 1)$.

Put $y(x) = \sum_{n=0}^{\infty} a_n x^n$ in the Legendre equation.

Equating the coefficient of x^n in the resulting equation, we get the recursive relation

$$(n+2)(n+1)a_{n+2} - n(n+1)a_n + p(p+1)a_n = 0, \quad n \geq 0$$

Legendre equation: Two independent solutions

$$\Longrightarrow \qquad a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)}\, a_n$$

Let us set $\boxed{a_0 = 1 \text{ and } a_1 = 0}$ in the recursion formula to find a first solution.

The solution is given by (note it is an even function)

$$y_1(x) := a_0 \left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \dots\right]$$

Let us find a second solution by setting $\boxed{a_0 = 0 \text{ and } a_1 = 1}$ in the recursion formula.

The second solution is given by (note it is an odd function)

$$y_2(x) := a_1 \left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \dots\right]$$

Legendre equation: Two independent solutions

Thus, the two independent solutions are

$$y_1(x) := a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \dots\right]$$

$$y_2(x) := a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \dots\right]$$

Remark

If $p \in \{0, 2, 4, \dots\} \cup \{-1, -3, -5, \dots\}$ then $y_1(x)$ is a polynomial function.

$y_2(x)$ is an odd function. If $p \in \{1, 3, 5, \dots\} \cup \{-2, -4, -6, \dots\}$ then $y_2(x)$ is a polynomial function.

Thus, if p is an integer then exactly one solution is a polynomial and the other is an infinite power series.

Legendre polynomials

The general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a Legendre function.

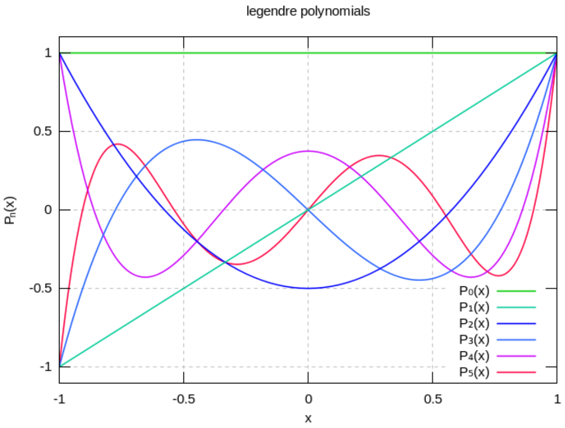
If $p = m$ is an integer, then precisely one of y_1 or y_2 is a polynomial, and it is called the m-th Legendre polynomial $P_m(x)$.

For $m \geq 0$ note that $P_m(x)$ is a polynomial of degree m. It is an even function if m is even and an odd function if m is odd.

Let us write down few Legendre polynomials.

$$\begin{array}{lllll}
P_0(x) & = & 1 & & \\
P_1(x) & = & x & & \\
P_2(x) & = & (1-3x^2)(\frac{-1}{2}) & = & \frac{1}{2}(3x^2-1) \\
P_3(x) & = & (x-\frac{5}{3}x^3)(\frac{-3}{2}) & = & \frac{1}{2}(5x^3-3x) \\
P_4(x) & = & (1-10x^2+\frac{35}{3}x^4)(\frac{3}{8}) & = & \frac{1}{8}(35x^4-30x^2+3) \\
P_5(x) & = & (x-\frac{14}{3}x^3+\frac{21}{5}x^5)(\frac{15}{8}) & = & \frac{1}{8}(63x^5-70x^3+15x)
\end{array}$$

The graphs of P_m's in the interval $(-1, 1)$ are given below.



What is so interesting about the collection of Legendre polynomials?

To answer this question we need some linear algebra.

Vector spaces

We will recall the notion of Inner product space from Linear Algebra.

First recall the notion of a vector space V over $\mathbb{R}$.

A vector space is a set equipped with two operations

- addition

$$v + w, \qquad v, w \in V$$

- scalar multiplication

$$cv, \qquad c \in \mathbb{R}, \quad v \in V$$

A vector space V has a dimension, which may not be finite.

Inner product spaces

Let V be a vector space over $\mathbb{R}$ (not necessarily finite-dimensional).

A bilinear form on V is a map

$$\langle , \rangle : V \times V \to \mathbb{R}$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for $a \in \mathbb{R}$ and $u, v \in V$.

An inner product on V is a bilinear form on V which is

- symmetric: $\langle v, w \rangle = \langle w, v \rangle$
- positive definite: $\langle v, v \rangle \geq 0$ for all v and $\langle v, v \rangle = 0$ iff $v = 0$

A vector space with an inner product is called an inner product space.

Orthogonality

In an inner product space V, two vectors u and v are orthogonal if $\langle u, v\rangle = 0$.

More generally, a set of vectors forms an orthogonal system if they are mutually orthogonal.

An orthogonal basis is an orthogonal system which is also a basis.

Example

Consider the vector space $\mathbb{R}^n$ with coordinate-wise addition and scalar multiplication. The rule

$$\langle (a_1, \dots, a_n), (b_1, \dots, b_n)\rangle := \sum_{i=1}^{n} a_i b_i$$

defines an inner product on $\mathbb{R}^n$.

The standard basis $\{e_1, \dots, e_n\}$ is an orthogonal basis of $\mathbb{R}^n$.

The previous example can be formulated more abstractly as follows.

Example

Let V be a finite-dimensional vector space with ordered basis $B = \{e_1, \ldots, e_n\}$.

For $u = \sum_{i=1}^{n} a_i e_i$ and $v = \sum_{i=1}^{n} b_i e_i$ define

$$\langle u, v \rangle := \sum_{i=1}^{n} a_i b_i$$

This defines an inner product on V

With this definition, $\{e_1, \ldots, e_n\}$ is an orthogonal basis of V.

Orthogonality

Lemma

Suppose V is a **finite** *dimensional inner product space, and $e_1, \ldots, e_n$ is an orthogonal basis.*

Then for any $v \in V$

$$v = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

Proof.

Write $v = \sum_{i=1}^{n} a_i e_i$. We want to find the coefficients a_j. Take inner product of v with e_j:

$$\langle v, e_j \rangle = \langle \sum_{i=1}^{n} a_i e_i, e_j \rangle = \sum_{i=1}^{n} a_i \langle e_i, e_j \rangle = a_j \langle e_j, e_j \rangle$$

Thus,

$$a_j = \frac{\langle v, e_j \rangle}{\langle e_j, e_j \rangle}$$

□

Orthogonal basis

Lemma

In a finite-dimensional inner product space, there always exists an orthogonal basis.

Start with any basis and modify it to an orthogonal basis by Gram-Schmidt orthogonalization.

This result is not necessarily true in infinite-dimensional inner product spaces.

For infinite dimensional vector spaces, we can only talk of a maximal orthogonal set.

A subset $\{e_1, e_2, \ldots\}$ is called a maximal orthogonal set for V if

- $\langle e_i, e_j \rangle = \delta_{ij}$
- $\langle v, e_i \rangle = 0$ for all i iff $v = 0$.

For a vector v in an inner product space, define

$$\|v\| := \langle v, v \rangle^{1/2}$$

This is called the norm or length of the vector v.

It satisfies the following three properties.

- $\|0\| = 0$ and $\|v\| > 0$ if $v \neq 0$
- $\|v + w\| \leq \|v\| + \|w\|$
- $\|av\| = |a|\|v\|$

for all $v, w \in V$ and $a \in \mathbb{R}$.

Pythagoras theorem

Theorem

For **orthogonal** *vectors* v *and* w *in any inner product space* V*,*

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Proof.

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= \langle v, v \rangle + \langle w, w \rangle \\ &= \|v\|^2 + \|w\|^2 \end{aligned}$$

□

More generally, for any orthogonal system $\{v_1, \ldots, v_n\}$

$$\|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2$$

The vector space of polynomials

The set of all polynomials in the variable x is a vector space denoted by $\mathcal{P}(x)$.

The set

$$\{1, x, x^2, \dots\}$$

is an infinite basis of the vector space $\mathcal{P}(x)$.

$\mathcal{P}(x)$ carries an inner product defined by

$$\langle f, g \rangle := \int_{-1}^{1} f(x) g(x) \, dx$$

We are integrating over finite interval $[-1, 1]$ which ensures that the integral is finite.

The norm of a polynomial is by definition $\langle f, f \rangle$

$$\|f\| := \left(\int_{-1}^{1} f(x) f(x) dx \right)^{1/2}$$

Derivative transfer

Note that

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

Integrating both sides we get

$$\int_{-1}^{1} \frac{d}{dx}(fg) = \int_{-1}^{1} g\frac{df}{dx} + \int_{-1}^{1} f\frac{dg}{dx}$$

$$\implies \qquad f(1)g(1) - f(-1)g(-1) = \int_{-1}^{1} g\frac{df}{dx} + \int_{-1}^{1} f\frac{dg}{dx}$$

Thus if

$$f(1)g(1) - f(-1)g(-1) = 0$$

then we get

$$\int_{-1}^{1} g\frac{df}{dx} = -\int_{-1}^{1} f\frac{dg}{dx}$$

This will be referred to as derivative-transfer

Orthogonality of Legendre polynomials

Since $P_m(x)$ is a polynomial of degree m, it follows that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

is a basis of the vector space of polynomials $\mathcal{P}(x)$.

Theorem

We have

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

i.e. Legendre polynomials form an orthogonal basis for the vector space $\mathcal{P}(x)$ and

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

Orthogonality of Legendre polynomials

The Legendre equation may be written as

$$((1-x^2)y')' + p(p+1)y = 0$$

In particular, $P_m(x)$ satisfies

$$((1-x^2)P_m'(x))' + m(m+1)P_m(x) = 0 \qquad (*)$$

Proof of Orthogonality.

Multiply $(*)$ by P_n and integrate to get

$$\int_{-1}^{1} ((1-x^2)P_m')'P_n + m(m+1)\int_{-1}^{1} P_m P_n = 0$$

By derivative transfer ($f = (1-x^2)P_m'$ and $g = P_n$), we get

$$-\int_{-1}^{1} (1-x^2)P_m'P_n' + m(m+1)\int_{-1}^{1} P_m P_n = 0$$

Orthogonality of Legendre polynomials

continued ...

Interchanging the roles of m and n, we get

$$-\int_{-1}^{1}(1-x^2)P_m' P_n' + n(n+1)\int_{-1}^{1} P_m P_n = 0$$

Subtracting the two identities, we obtain

$$[m(m+1) - n(n+1)]\int_{-1}^{1} P_m P_n = 0$$

If $m \neq n$ we get

$$\int_{-1}^{1} P_m P_n = 0$$

Thus, P_m and P_n are orthogonal. □

It only remains to show that $\|P_n(x)\|^2 = \dfrac{2}{2n+1}$.

We need some intermediate steps before we can show this.

Denote by D the differential operator $\frac{d}{dx}$.

Let us first note that for $0 \leq i < n$

$$\big(D^i(x^2-1)^n\big)(1) = 0$$

This is clear once we observe

$$\begin{aligned} D^i(x^2-1)^n &= D^i\big((x-1)^n(2+x-1)^n\big) \\ &= D^i\big(2^n(x-1)^n + (*)(x-1)^{n+1} + \cdots\big) \end{aligned}$$

By the same reasoning we get for $0 \leq i < n$

$$\big(D^i(x^2-1)^n\big)(-1) = 0$$

Consider the polynomial of degree n given by

$$y(x) = D^n(x^2-1)^n$$

For $k < n$ consider the integral

$$\int_{-1}^{1} P_k(x)y(x) = \int_{-1}^{1} P_k(x)D(D^{n-1}(x^2-1)^n)$$

applying derivative transfer with $f = D^{n-1}(x^2-1)^n$ and $g = P_k(x)$ we get

$$\begin{aligned}\int_{-1}^{1} P_k(x)y(x) &= -\int_{-1}^{1} DP_k(x)D^{n-1}(x^2-1)^n \\ &= \int_{-1}^{1} D^2P_k(x)D^{n-2}(x^2-1)^n \\ &= \int_{-1}^{1} D^nP_k(x)(x^2-1)^n = 0\end{aligned}$$

We have repeatedly applied derivative transfer with $f = D^{n-i}(x^2-1)^n$ and $g = D^{i-1}P_k(x)$.

Since $P_k(x)$ is a polynomial of degree k we get that $D^nP_k(x) = 0$.

This forces that $y(x) = cP_n(x)$ for some nonzero constant c as we know that $P_k(x)$'s are orthogonal to each other.

$$\begin{aligned} D^n(x^2-1)^n &= D^n\big((x-1)^n(2+x-1)^n\big) \\ &= D^n\big(2^n(x-1)^n + (*)(x-1)^{n+1} + \cdots\big) \end{aligned}$$

From the above it is clear that

$$y(1) = n!2^n$$

Thus, we can normalize our Legendre polynomials so that $P_m(1) = 1$. That is, take

$$P_m(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

This is called Rodrigues formula.

Proof.

$$\int_{-1}^{1} P_n(x)P_n(x)\,dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\,dx$$

by derivative transfer

$$= \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx$$

$$I_n = \int_{-1}^{1}(1-x^2)^n\,dx = \int_{-1}^{1}(1-x^2)^n\frac{dx}{dx}dx$$

$$\overset{dt}{=} 2n\int_{-1}^{1}(1-x^2)^{n-1}x^2dx = -2nI_n + 2nI_{n-1}$$

Computing $\|P_n(x)\|$

Proof.

We get the recursive relation

$$(2n+1)I_n = 2nI_{n-1}$$

We conclude that

$$I_n = \frac{2n}{2n+1}\frac{2(n-1)}{2n-1}\cdots\frac{2}{3}I_0$$

We conclude that

$$\|P_n(x)\| = \frac{(2n)!}{2^{2n}(n!)^2}\frac{2n}{2n+1}\frac{2(n-1)}{2n-1}\cdots\frac{2}{3}I_0$$
$$= \frac{I_0}{2n+1} = \frac{2}{2n+1}$$

□

This completes the proof of the theorem.

Exercises

These exercises are related to some facts from linear algebra that we used in the lecture today.

1. Recall the proof of the Gram Schmidt orthogonalization lemma.
2. Let $f_i(x)$ (for $i \geq 0$) be a collection of nonzero polynomials. Assume that $f_i(x)$ has degree i. Show that $\{f_0(x), f_1(x), \ldots, f_n(x)\}$ is a basis for the vector space consisting of polynomials of degree $\leq n$.

Recall

The Legendre equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

When $p = m$ is an integer then exactly one of the two independent solutions is a polynomial, denoted by $P_m(x)$. This is a polynomial of degree m.

Theorem

We have

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \textit{if } m \neq n \\ \frac{2}{2n+1} & \textit{if } m = n \end{cases}$$

i.e. Legendre polynomials form an orthogonal basis for the vector space $\mathcal{P}(x)$ and

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

Expansion of polynomial in terms of P_n's

Since each $P_n(x)$ is a polynomial of degree n, we see that

$$\{P_0(x), P_1(x), P_2(x), \ldots\}$$

form a basis for the vector space of polynomials $\mathcal{P}(x)$.
If $f(x)$ is a polynomial of degree n, then we can express

$$f(x) = \sum_{k=0}^{n} a_k P_k(x) \qquad a_k \in \mathbb{R}$$

To find a_k, we can use orthogonality of P_n's.

$$\int_{-1}^{1} f(x) P_k(x)\, dx = \int_{-1}^{1} \left(\sum_{i=0}^{n} a_i P_i(x) \right) P_k(x)\, dx$$

$$= \sum_{i=0}^{n} \left(\int_{-1}^{1} a_i P_i(x) P_k(x)\, dx \right) = a_k \int_{-1}^{1} P_k(x) P_k(x)\, dx$$

$$\implies a_k = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_k(x)\, dx$$

Square-integrable functions

A function $f(x)$ on $[-1,1]$ is square-integrable if

$$\int_{-1}^{1} f(x)f(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable.

The set of all square-integrable functions on $[-1,1]$ is a vector space and is denoted by $L^2([-1,1])$.

For square-integrable functions f and g, we define their inner product by

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx$$

The Legendre polynomials no longer form a basis for the vector space $L^2([-1,1])$ of square-integrable functions.

In fact, the space $L^2([-1,1])$ is HUGE in comparison with the space spanned by the Legendre polynomials.

What do we mean by HUGE here?

The set of rational numbers ($\mathbb{Q}$) is contained in the set of real numbers ($\mathbb{R}$) and is "dense". However, the size of $\mathbb{R}$ is much larger than the size of $\mathbb{Q}$. These ideas were formulated precisely by Georg Cantor.

But Legendre polynomials form a maximal orthogonal set in $L^2([-1,1])$.

This means that a square-integrable function which is orthogonal to all Legendre polynomials is necessarily the constant function "0" (a nontrivial fact).

We can expand any square-integrable function $f(x)$ on $[-1,1]$ in a series of Legendre polynomials

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

This is called the Fourier-Legendre series (or simply the Legendre series) of $f(x)$.

Fourier-Legendre series

Theorem (Convergence in norm)

The Fourier-Legendre series of $f(x) \in L^2([-1,1])$ given by

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

converges in L^2 norm to $f(x)$, that is

$$\|f(x) - \sum_{n=0}^{m} c_n P_n(x)\| \to 0 \quad \text{as} \quad m \to \infty$$

Pointwise convergence of Fourier-Legendre series to $f(x)$ is more delicate.

There are two issues here:

- Does the Fourier-Legendre series converge at x?
- If yes, then does it converge to $f(x)$?

Fourier-Legendre series

A useful result in this direction is the Legendre expansion theorem:

Theorem

If both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[-1, 1]$,
then the Legendre series converges to

$$\frac{1}{2}(f(x_-) + f(x_+)) \quad \text{for } -1 < x < 1$$

$$f(-1_+) \quad \text{for } x = -1$$

$$f(1_-) \quad \text{for } x = 1$$

In particular, the series converges to $f(x)$ at every point of continuity x.

Example

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

The Legendre series of $f(x)$ is

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

Since $P_{2n}(x)$ is even function and f is an odd function, we get

$$c_{2n} = 0 \qquad n \geq 0$$

Recall, $P_1(x) = x$, so

$$c_1 = \frac{3}{2} \int_{-1}^{1} f(x) x\, dx = \frac{3}{2}$$

Example (continued ...)

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$, so

$$c_3 = \frac{7}{2}\int_{-1}^{1} f(x)\frac{1}{2}(5x^3 - 3x)\,dx = \frac{7}{2}(\frac{5}{4}x^4 - \frac{3}{2}x^2)\Big|_0^1 = -\frac{7}{8}$$

Check that the Legendre series of f is

$$\frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \dots$$

By the Legendre expansion theorem, this series converges to $f(x)$ for $x \neq 0$ and to 0 for $x = 0$.

Least Square Approximation

Theorem

Suppose we want to approximate $f \in L^2([-1,1])$ in the sense of least square by polynomials $p(x)$ of degree $\leq n$; that is, we want to find a polynomial $p(x)$ which minimizes

$$I = \int_{-1}^{1} [f(x) - p(x)]^2 \, dx$$

Then the minimizing polynomial is precisely the first $n+1$ terms of the Legendre series of $f(x)$, i.e.

$$c_0 P_0(x) + \ldots + c_n P_n(x) \qquad c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

Proof.

Write degree $\leq n$ polynomial $p(x) = \sum_{k=0}^{n} b_k P_k(x)$, then

$$\begin{aligned} I &= \int_{-1}^{1} \left[f(x) - \sum_{k=0}^{n} b_k P_k(x) \right]^2 \, dx \\ &= \int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^{n} b_k \left[\int_{-1}^{1} f(x) P_k(x) \, dx \right] \\ &= \int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^{n} b_k \frac{2c_k}{2k+1} \\ &= \int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2}{2k+1} (b_k - c_k)^2 - \sum_{k=0}^{n} \frac{2}{2k+1} c_k^2 \end{aligned}$$

Clearly, I is minimum when $b_k = c_k$ for $k = 0, \ldots, n$.

Caution. If f has a power series expansion on $[-1, 1]$, then best "least square polynomial approximation" to $f(x)$ is not the partial sums of the power series, in general.

Some remarks

This brings to an end the discussion of second order linear ODE's which we can solve by power series.

Before we go on to more complicated ODE's, let us review what we have done so far.

1. Given an ODE of the type

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \qquad (*)$$

first convert it to the standard form

$$y'' + \frac{F_1(x)}{F_0(x)}y' + \frac{F_2(x)}{F_0(x)}y = 0 \qquad (**)$$

Let

$$p(x) := \frac{F_1(x)}{F_0(x)} \qquad q(x) := \frac{F_2(x)}{F_0(x)}$$

Some remarks

2. Now find the set

$$U := \{x_0 \in \mathbb{R} \,|\, p(x), q(x) \text{ are analytic at } x_0\}$$

3. By the existence theorem, for every $x_0 \in U$, there will exist two independent solutions to the above ODE, call them $y_1(x)$ and $y_2(x)$, such that both of them will be analytic in an interval I around x_0.

4. To find the solutions in a neighborhood of x_0, set $y(x) = \sum_{n \geq 0} a_n (x - x_0)^n$ into the ODE $(*)$ or $(**)$ and get recursive relations involving the a_n. Note that when you do this, the coefficient functions $(p(x), q(x), F_0(x), ..)$ have to be written as power series in $x - x_0$. Note that the recursive relation you get, will be same, irrespective of whether you choose equation $(*)$ or $(**)$.

5. Thus, depending on the situation, you may want to choose $(*)$ or $(**)$.
For example, the Legendre equation, in the open interval $(-1, 1)$ around $x_0 = 0$, the equation $(*)$ looks like

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

while $(**)$ looks like

$$y'' - 2\Big(\sum_{n\geq 0} x^{2n+1}\Big)y' + p(p + 1)\Big(\sum_{n\geq 0} x^{2n}\Big)y = 0$$

In this case it is clear that, we should choose $(*)$, as it will be easier to work with. This is what we did in class.