

# MA-207 Differential Equations II

Ronnie Sebastian



Department of Mathematics  
Indian Institute of Technology Bombay  
Powai, Mumbai - 76

September 7, 2021

The important things we did so far are

The important things we did so far are

- 1 How to compute the radius of convergence of a power series

The important things we did so far are

- 1 How to compute the radius of convergence of a power series
- 2 Power series defines a nice function in its interval of convergence

The important things we did so far are

- 1 How to compute the radius of convergence of a power series
- 2 Power series defines a nice function in its interval of convergence
- 3 Suppose we are given an ODE:  $y'' + p(x)y' + q(x)y = 0$ , and  $p(x)$  and  $q(x)$  are analytic (given by power series) in an interval  $I$  around  $x_0$ , then the solution  $y$  is also analytic on  $I$ .

The important things we did so far are

- 1 How to compute the radius of convergence of a power series
- 2 Power series defines a nice function in its interval of convergence
- 3 Suppose we are given an ODE:  $y'' + p(x)y' + q(x)y = 0$ , and  $p(x)$  and  $q(x)$  are analytic (given by power series) in an interval  $I$  around  $x_0$ , then the solution  $y$  is also analytic on  $I$ .
- 4 We can compute the two independent solutions, to an ODE as above, by plugging in a power series into the ODE and getting recursive relation for coefficients.



# Legendre equation

The following ODE is known as the **Legendre equation**.

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

Here  $p$  denotes a fixed real number.



# Legendre equation

The following ODE is known as the **Legendre equation**.

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

Here  $p$  denotes a fixed real number.

By Existence theorem, power series solution in  $x$  exists on the interval  $(-1, 1)$ .

# Legendre equation

The following ODE is known as the **Legendre equation**.

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

Here  $p$  denotes a fixed real number.

By Existence theorem, power series solution in  $x$  exists on the interval  $(-1, 1)$ .

Put  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  in the Legendre equation.

The following ODE is known as the **Legendre equation**.

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

Here  $p$  denotes a fixed real number.

By Existence theorem, power series solution in  $x$  exists on the interval  $(-1, 1)$ .

Put  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  in the Legendre equation.

Equating the coefficient of  $x^n$  in the resulting equation, we get the recursive relation

$$(n + 2)(n + 1)a_{n+2} - n(n + 1)a_n + p(p + 1)a_n = 0, \quad n \geq 0$$

## Legendre equation: Two independent solutions

$$\implies a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)} a_n$$

## Legendre equation: Two independent solutions

$$\implies a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)} a_n$$

Let us set  $a_0 = 1$  and  $a_1 = 0$  in the recursion formula to find a first solution.

## Legendre equation: Two independent solutions

$$\implies a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)} a_n$$

Let us set  $a_0 = 1$  and  $a_1 = 0$  in the recursion formula to find a first solution.

The solution is given by (note it is an even function)

$$y_1(x) := a_0 \left[ 1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p+1)(p-2)(p+3)}{4!} x^4 + \dots \right]$$

## Legendre equation: Two independent solutions

$$\implies a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)} a_n$$

Let us set  $a_0 = 1$  and  $a_1 = 0$  in the recursion formula to find a first solution.

The solution is given by (note it is an even function)

$$y_1(x) := a_0 \left[ 1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p+1)(p-2)(p+3)}{4!} x^4 + \dots \right]$$

Let us find a second solution by setting  $a_0 = 0$  and  $a_1 = 1$  in the recursion formula.

## Legendre equation: Two independent solutions

$$\implies a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)} a_n$$

Let us set  $a_0 = 1$  and  $a_1 = 0$  in the recursion formula to find a first solution.

The solution is given by (note it is an even function)

$$y_1(x) := a_0 \left[ 1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p+1)(p-2)(p+3)}{4!} x^4 + \dots \right]$$

Let us find a second solution by setting  $a_0 = 0$  and  $a_1 = 1$  in the recursion formula.

The second solution is given by (note it is an odd function)

$$y_2(x) := a_1 \left[ x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!} x^5 + \dots \right]$$



# Legendre equation: Two independent solutions

Thus, the two independent solutions are

# Legendre equation: Two independent solutions

Thus, the two independent solutions are

$$y_1(x) := a_0 \left[ 1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \dots \right]$$

$$y_2(x) := a_1 \left[ x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \dots \right]$$

# Legendre equation: Two independent solutions

Thus, the two independent solutions are

$$y_1(x) := a_0 \left[ 1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \dots \right]$$

$$y_2(x) := a_1 \left[ x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \dots \right]$$

## Remark

If  $p \in \{0, 2, 4, \dots\} \cup \{-1, -3, -5, \dots\}$  then  $y_1(x)$  is a polynomial function.

# Legendre equation: Two independent solutions

Thus, the two independent solutions are

$$y_1(x) := a_0 \left[ 1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \dots \right]$$

$$y_2(x) := a_1 \left[ x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \dots \right]$$

## Remark

If  $p \in \{0, 2, 4, \dots\} \cup \{-1, -3, -5, \dots\}$  then  $y_1(x)$  is a polynomial function.

$y_2(x)$  is an odd function. If  $p \in \{1, 3, 5, \dots\} \cup \{-2, -4, -6, \dots\}$  then  $y_2(x)$  is a polynomial function.

# Legendre equation: Two independent solutions

Thus, the two independent solutions are

$$y_1(x) := a_0 \left[ 1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \dots \right]$$

$$y_2(x) := a_1 \left[ x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \dots \right]$$

## Remark

If  $p \in \{0, 2, 4, \dots\} \cup \{-1, -3, -5, \dots\}$  then  $y_1(x)$  is a polynomial function.

$y_2(x)$  is an odd function. If  $p \in \{1, 3, 5, \dots\} \cup \{-2, -4, -6, \dots\}$  then  $y_2(x)$  is a polynomial function.

Thus, if  $p$  is an integer then exactly one solution is a polynomial and the other is an infinite power series.

# Legendre polynomials

The general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a **Legendre function**.

# Legendre polynomials

The general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a **Legendre function**.

If  $p = m$  is an integer, then precisely one of  $y_1$  or  $y_2$  is a polynomial, and it is called the  **$m$ -th Legendre polynomial**  $P_m(x)$ .

# Legendre polynomials

The general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a **Legendre function**.

If  $p = m$  is an integer, then precisely one of  $y_1$  or  $y_2$  is a polynomial, and it is called the  **$m$ -th Legendre polynomial  $P_m(x)$** .

For  $m \geq 0$  note that  $P_m(x)$  is a polynomial of degree  $m$ . It is an even function if  $m$  is even and an odd function if  $m$  is odd.



# Legendre polynomials

The general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a **Legendre function**.

If  $p = m$  is an integer, then precisely one of  $y_1$  or  $y_2$  is a polynomial, and it is called the  $m$ -th **Legendre polynomial**  $P_m(x)$ .

For  $m \geq 0$  note that  $P_m(x)$  is a polynomial of degree  $m$ . It is an even function if  $m$  is even and an odd function if  $m$  is odd.

Let us write down few **Legendre polynomials**.

# Legendre polynomials

The general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a **Legendre function**.

If  $p = m$  is an integer, then precisely one of  $y_1$  or  $y_2$  is a polynomial, and it is called the  $m$ -th **Legendre polynomial**  $P_m(x)$ .

For  $m \geq 0$  note that  $P_m(x)$  is a polynomial of degree  $m$ . It is an even function if  $m$  is even and an odd function if  $m$  is odd.

Let us write down few **Legendre polynomials**.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = (1 - 3x^2)\left(\frac{-1}{2}\right) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \left(x - \frac{5}{3}x^3\right)\left(\frac{-3}{2}\right) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \left(1 - 10x^2 + \frac{35}{3}x^4\right)\left(\frac{3}{8}\right) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

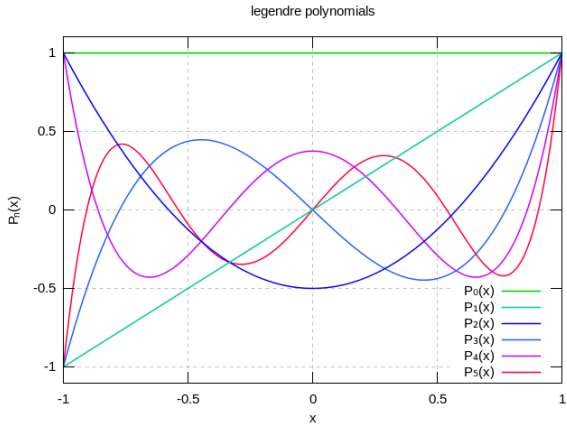
$$P_5(x) = \left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right)\left(\frac{15}{8}\right) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

# Legendre polynomials

The graphs of  $P_m$ 's in the interval  $(-1, 1)$  are given below.

# Legendre polynomials

The graphs of  $P_m$ 's in the interval  $(-1, 1)$  are given below.



What is so interesting about the collection of Legendre polynomials?

## What is so interesting about the collection of Legendre polynomials?

To answer this question we need some linear algebra.

We will recall the notion of Inner product space from Linear Algebra.

We will recall the notion of Inner product space from Linear Algebra.

First recall the notion of a **vector space**  $V$  over  $\mathbb{R}$ .



We will recall the notion of Inner product space from Linear Algebra.

First recall the notion of a **vector space**  $V$  over  $\mathbb{R}$ .

A vector space is a set equipped with two operations

- addition

$$v + w, \quad v, w \in V$$

- scalar multiplication

$$cv, \quad c \in \mathbb{R}, \quad v \in V$$

We will recall the notion of Inner product space from Linear Algebra.

First recall the notion of a **vector space**  $V$  over  $\mathbb{R}$ .

A vector space is a set equipped with two operations

- addition

$$v + w, \quad v, w \in V$$

- scalar multiplication

$$cv, \quad c \in \mathbb{R}, \quad v \in V$$

A vector space  $V$  has a dimension, which may not be finite.

# Inner product spaces

Let  $V$  be a vector space over  $\mathbb{R}$  (not necessarily finite-dimensional).

# Inner product spaces

Let  $V$  be a vector space over  $\mathbb{R}$  (not necessarily finite-dimensional).

A **bilinear form** on  $V$  is a map

$$\langle, \rangle : V \times V \rightarrow \mathbb{R}$$

# Inner product spaces

Let  $V$  be a vector space over  $\mathbb{R}$  (not necessarily finite-dimensional).

A **bilinear form** on  $V$  is a map

$$\langle, \rangle : V \times V \rightarrow \mathbb{R}$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for  $a \in \mathbb{R}$  and  $u, v \in V$ .

# Inner product spaces

Let  $V$  be a vector space over  $\mathbb{R}$  (not necessarily finite-dimensional).

A **bilinear form** on  $V$  is a map

$$\langle, \rangle : V \times V \rightarrow \mathbb{R}$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for  $a \in \mathbb{R}$  and  $u, v \in V$ .

An **inner product** on  $V$  is a bilinear form on  $V$  which is

# Inner product spaces

Let  $V$  be a vector space over  $\mathbb{R}$  (not necessarily finite-dimensional).

A **bilinear form** on  $V$  is a map

$$\langle, \rangle : V \times V \rightarrow \mathbb{R}$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for  $a \in \mathbb{R}$  and  $u, v \in V$ .

An **inner product** on  $V$  is a bilinear form on  $V$  which is

- symmetric:  $\langle v, w \rangle = \langle w, v \rangle$

# Inner product spaces

Let  $V$  be a vector space over  $\mathbb{R}$  (not necessarily finite-dimensional).

A **bilinear form** on  $V$  is a map

$$\langle, \rangle : V \times V \rightarrow \mathbb{R}$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for  $a \in \mathbb{R}$  and  $u, v \in V$ .

An **inner product** on  $V$  is a bilinear form on  $V$  which is

- symmetric:  $\langle v, w \rangle = \langle w, v \rangle$
- positive definite:  $\langle v, v \rangle \geq 0$  for all  $v$  and  $\langle v, v \rangle = 0$  iff  $v = 0$



# Inner product spaces

Let  $V$  be a vector space over  $\mathbb{R}$  (not necessarily finite-dimensional).

A **bilinear form** on  $V$  is a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for  $a \in \mathbb{R}$  and  $u, v \in V$ .

An **inner product** on  $V$  is a bilinear form on  $V$  which is

- symmetric:  $\langle v, w \rangle = \langle w, v \rangle$
- positive definite:  $\langle v, v \rangle \geq 0$  for all  $v$  and  $\langle v, v \rangle = 0$  iff  $v = 0$

A vector space with an inner product is called an **inner product space**.



# Orthogonality

In an inner product space  $V$ , two vectors  $u$  and  $v$  are **orthogonal** if  $\langle u, v \rangle = 0$ .

# Orthogonality

In an inner product space  $V$ , two vectors  $u$  and  $v$  are **orthogonal** if  $\langle u, v \rangle = 0$ .

More generally, a set of vectors forms an **orthogonal system** if they are mutually orthogonal.

# Orthogonality

In an inner product space  $V$ , two vectors  $u$  and  $v$  are **orthogonal** if  $\langle u, v \rangle = 0$ .

More generally, a set of vectors forms an **orthogonal system** if they are mutually orthogonal.

An **orthogonal basis** is an orthogonal system which is also a basis.

# Orthogonality

In an inner product space  $V$ , two vectors  $u$  and  $v$  are **orthogonal** if  $\langle u, v \rangle = 0$ .

More generally, a set of vectors forms an **orthogonal system** if they are mutually orthogonal.

An **orthogonal basis** is an orthogonal system which is also a basis.

## Example

Consider the vector space  $\mathbb{R}^n$  with coordinate-wise addition and scalar multiplication.

# Orthogonality

In an inner product space  $V$ , two vectors  $u$  and  $v$  are **orthogonal** if  $\langle u, v \rangle = 0$ .

More generally, a set of vectors forms an **orthogonal system** if they are mutually orthogonal.

An **orthogonal basis** is an orthogonal system which is also a basis.

## Example

Consider the vector space  $\mathbb{R}^n$  with coordinate-wise addition and scalar multiplication. The rule

$$\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle := \sum_{i=1}^n a_i b_i$$

defines an inner product on  $\mathbb{R}^n$ .

# Orthogonality

In an inner product space  $V$ , two vectors  $u$  and  $v$  are **orthogonal** if  $\langle u, v \rangle = 0$ .

More generally, a set of vectors forms an **orthogonal system** if they are mutually orthogonal.

An **orthogonal basis** is an orthogonal system which is also a basis.

## Example

Consider the vector space  $\mathbb{R}^n$  with coordinate-wise addition and scalar multiplication. The rule

$$\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle := \sum_{i=1}^n a_i b_i$$

defines an inner product on  $\mathbb{R}^n$ .

The standard basis  $\{e_1, \dots, e_n\}$  is an orthogonal basis of  $\mathbb{R}^n$ .



The previous example can be formulated more abstractly as follows.

The previous example can be formulated more abstractly as follows.

## Example

Let  $V$  be a finite-dimensional vector space with ordered basis  $B = \{e_1, \dots, e_n\}$ .

The previous example can be formulated more abstractly as follows.

## Example

Let  $V$  be a finite-dimensional vector space with ordered basis  $B = \{e_1, \dots, e_n\}$ .

For  $u = \sum_{i=1}^n a_i e_i$  and  $v = \sum_{i=1}^n b_i e_i$  define

$$\langle u, v \rangle := \sum_{i=1}^n a_i b_i$$

The previous example can be formulated more abstractly as follows.

## Example

Let  $V$  be a finite-dimensional vector space with ordered basis  $B = \{e_1, \dots, e_n\}$ .

For  $u = \sum_{i=1}^n a_i e_i$  and  $v = \sum_{i=1}^n b_i e_i$  define

$$\langle u, v \rangle := \sum_{i=1}^n a_i b_i$$

This defines an inner product on  $V$

The previous example can be formulated more abstractly as follows.

## Example

Let  $V$  be a finite-dimensional vector space with ordered basis  $B = \{e_1, \dots, e_n\}$ .

For  $u = \sum_{i=1}^n a_i e_i$  and  $v = \sum_{i=1}^n b_i e_i$  define

$$\langle u, v \rangle := \sum_{i=1}^n a_i b_i$$

This defines an inner product on  $V$

With this definition,  $\{e_1, \dots, e_n\}$  is an orthogonal basis of  $V$ .

## Lemma

Suppose  $V$  is a **finite** dimensional inner product space, and  $e_1, \dots, e_n$  is an orthogonal basis.

## Lemma

Suppose  $V$  is a **finite** dimensional inner product space, and  $e_1, \dots, e_n$  is an orthogonal basis.

Then for any  $v \in V$

$$v = \sum_{i=1}^n \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

## Lemma

Suppose  $V$  is a **finite** dimensional inner product space, and  $e_1, \dots, e_n$  is an orthogonal basis.

Then for any  $v \in V$

$$v = \sum_{i=1}^n \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

## Proof.

Write  $v = \sum_{i=1}^n a_i e_i$ .



## Lemma

Suppose  $V$  is a **finite** dimensional inner product space, and  $e_1, \dots, e_n$  is an orthogonal basis.

Then for any  $v \in V$

$$v = \sum_{i=1}^n \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

## Proof.

Write  $v = \sum_{i=1}^n a_i e_i$ . We want to find the coefficients  $a_j$ .

## Lemma

Suppose  $V$  is a **finite** dimensional inner product space, and  $e_1, \dots, e_n$  is an orthogonal basis.

Then for any  $v \in V$

$$v = \sum_{i=1}^n \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

## Proof.

Write  $v = \sum_{i=1}^n a_i e_i$ . We want to find the coefficients  $a_j$ . Take inner product of  $v$  with  $e_j$ :

## Lemma

Suppose  $V$  is a **finite** dimensional inner product space, and  $e_1, \dots, e_n$  is an orthogonal basis.

Then for any  $v \in V$

$$v = \sum_{i=1}^n \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

## Proof.

Write  $v = \sum_{i=1}^n a_i e_i$ . We want to find the coefficients  $a_j$ . Take inner product of  $v$  with  $e_j$ :

$$\langle v, e_j \rangle = \left\langle \sum_{i=1}^n a_i e_i, e_j \right\rangle = \sum_{i=1}^n a_i \langle e_i, e_j \rangle = a_j \langle e_j, e_j \rangle$$

## Lemma

Suppose  $V$  is a **finite** dimensional inner product space, and  $e_1, \dots, e_n$  is an orthogonal basis.

Then for any  $v \in V$

$$v = \sum_{i=1}^n \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

## Proof.

Write  $v = \sum_{i=1}^n a_i e_i$ . We want to find the coefficients  $a_j$ . Take inner product of  $v$  with  $e_j$ :

$$\langle v, e_j \rangle = \left\langle \sum_{i=1}^n a_i e_i, e_j \right\rangle = \sum_{i=1}^n a_i \langle e_i, e_j \rangle = a_j \langle e_j, e_j \rangle$$

Thus, 
$$a_j = \frac{\langle v, e_j \rangle}{\langle e_j, e_j \rangle}$$



## Lemma

*In a finite-dimensional inner product space, there always exists an orthogonal basis.*

## Lemma

*In a finite-dimensional inner product space, there always exists an orthogonal basis.*

Start with any basis and modify it to an orthogonal basis by **Gram-Schmidt orthogonalization**.

## Lemma

*In a finite-dimensional inner product space, there always exists an orthogonal basis.*

Start with any basis and modify it to an orthogonal basis by **Gram-Schmidt orthogonalization**.

This result is not necessarily true in infinite-dimensional inner product spaces.

## Lemma

*In a finite-dimensional inner product space, there always exists an orthogonal basis.*

Start with any basis and modify it to an orthogonal basis by **Gram-Schmidt orthogonalization**.

This result is not necessarily true in infinite-dimensional inner product spaces.

For infinite dimensional vector spaces, we can only talk of a **maximal orthogonal set**.



## Lemma

*In a finite-dimensional inner product space, there always exists an orthogonal basis.*

Start with any basis and modify it to an orthogonal basis by **Gram-Schmidt orthogonalization**.

This result is not necessarily true in infinite-dimensional inner product spaces.

For infinite dimensional vector spaces, we can only talk of a **maximal orthogonal set**.

A subset  $\{e_1, e_2, \dots\}$  is called a maximal orthogonal set for  $V$  if

## Lemma

*In a finite-dimensional inner product space, there always exists an orthogonal basis.*

Start with any basis and modify it to an orthogonal basis by **Gram-Schmidt orthogonalization**.

This result is not necessarily true in infinite-dimensional inner product spaces.

For infinite dimensional vector spaces, we can only talk of a **maximal orthogonal set**.

A subset  $\{e_1, e_2, \dots\}$  is called a maximal orthogonal set for  $V$  if

- $\langle e_i, e_j \rangle = \delta_{ij}$

## Lemma

*In a finite-dimensional inner product space, there always exists an orthogonal basis.*

Start with any basis and modify it to an orthogonal basis by **Gram-Schmidt orthogonalization**.

This result is not necessarily true in infinite-dimensional inner product spaces.

For infinite dimensional vector spaces, we can only talk of a **maximal orthogonal set**.

A subset  $\{e_1, e_2, \dots\}$  is called a maximal orthogonal set for  $V$  if

- $\langle e_i, e_j \rangle = \delta_{ij}$
- $\langle v, e_i \rangle = 0$  for all  $i$  iff  $v = 0$ .

# Length of a vector

For a vector  $v$  in an inner product space, define

$$\|v\| := \langle v, v \rangle^{1/2}$$

# Length of a vector

For a vector  $v$  in an inner product space, define

$$\|v\| := \langle v, v \rangle^{1/2}$$

This is called the **norm** or **length** of the vector  $v$ .

# Length of a vector

For a vector  $v$  in an inner product space, define

$$\|v\| := \langle v, v \rangle^{1/2}$$

This is called the **norm** or **length** of the vector  $v$ .

It satisfies the following three properties.

# Length of a vector

For a vector  $v$  in an inner product space, define

$$\|v\| := \langle v, v \rangle^{1/2}$$

This is called the **norm** or **length** of the vector  $v$ .

It satisfies the following three properties.

- $\|0\| = 0$  and  $\|v\| > 0$  if  $v \neq 0$
- $\|v + w\| \leq \|v\| + \|w\|$
- $\|av\| = |a|\|v\|$

for all  $v, w \in V$  and  $a \in \mathbb{R}$ .



# Pythagoras theorem

## Theorem

For **orthogonal** vectors  $v$  and  $w$  in any inner product space  $V$ ,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

## Theorem

For **orthogonal** vectors  $v$  and  $w$  in any inner product space  $V$ ,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

## Proof.

## Theorem

For **orthogonal** vectors  $v$  and  $w$  in any inner product space  $V$ ,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

## Proof.

$$\|v + w\|^2 = \langle v + w, v + w \rangle$$

## Theorem

For **orthogonal** vectors  $v$  and  $w$  in any inner product space  $V$ ,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

## Proof.

$$\begin{aligned}\|v + w\|^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle\end{aligned}$$

## Theorem

For **orthogonal** vectors  $v$  and  $w$  in any inner product space  $V$ ,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

## Proof.

$$\begin{aligned}\|v + w\|^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= \langle v, v \rangle + \langle w, w \rangle\end{aligned}$$

## Theorem

For **orthogonal** vectors  $v$  and  $w$  in any inner product space  $V$ ,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

## Proof.

$$\begin{aligned}\|v + w\|^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= \langle v, v \rangle + \langle w, w \rangle \\ &= \|v\|^2 + \|w\|^2\end{aligned}$$

□

## Theorem

For **orthogonal** vectors  $v$  and  $w$  in any inner product space  $V$ ,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

## Proof.

$$\begin{aligned}\|v + w\|^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= \langle v, v \rangle + \langle w, w \rangle \\ &= \|v\|^2 + \|w\|^2\end{aligned}$$



More generally, for any orthogonal system  $\{v_1, \dots, v_n\}$

$$\|v_1 + \dots + v_n\|^2 = \|v_1\|^2 + \dots + \|v_n\|^2$$



# The vector space of polynomials

# The vector space of polynomials

The set of all polynomials in the variable  $x$  is a vector space denoted by  $\mathcal{P}(x)$ .

# The vector space of polynomials

The set of all polynomials in the variable  $x$  is a vector space denoted by  $\mathcal{P}(x)$ .

The set

$$\{1, x, x^2, \dots\}$$

is an infinite basis of the vector space  $\mathcal{P}(x)$ .

# The vector space of polynomials

The set of all polynomials in the variable  $x$  is a vector space denoted by  $\mathcal{P}(x)$ .

The set

$$\{1, x, x^2, \dots\}$$

is an infinite basis of the vector space  $\mathcal{P}(x)$ .

$\mathcal{P}(x)$  carries an inner product defined by

$$\langle f, g \rangle := \int_{-1}^1 f(x)g(x) dx$$

# The vector space of polynomials

The set of all polynomials in the variable  $x$  is a vector space denoted by  $\mathcal{P}(x)$ .

The set

$$\{1, x, x^2, \dots\}$$

is an infinite basis of the vector space  $\mathcal{P}(x)$ .

$\mathcal{P}(x)$  carries an inner product defined by

$$\langle f, g \rangle := \int_{-1}^1 f(x)g(x) dx$$

We are integrating over finite interval  $[-1, 1]$  which ensures that the integral is finite.

# The vector space of polynomials

The set of all polynomials in the variable  $x$  is a vector space denoted by  $\mathcal{P}(x)$ .

The set

$$\{1, x, x^2, \dots\}$$

is an infinite basis of the vector space  $\mathcal{P}(x)$ .

$\mathcal{P}(x)$  carries an inner product defined by

$$\langle f, g \rangle := \int_{-1}^1 f(x)g(x) dx$$

We are integrating over finite interval  $[-1, 1]$  which ensures that the integral is finite.

The **norm of a polynomial** is by definition  $\langle f, f \rangle$

$$\|f\| := \left( \int_{-1}^1 f(x)f(x) dx \right)^{1/2}$$

# Derivative transfer

Note that

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

# Derivative transfer

Note that

$$\frac{d}{dx}(fg) = g \frac{df}{dx} + f \frac{dg}{dx}$$

Integrating both sides we get

$$\int_{-1}^1 \frac{d}{dx}(fg) = \int_{-1}^1 g \frac{df}{dx} + \int_{-1}^1 f \frac{dg}{dx}$$



# Derivative transfer

Note that

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

Integrating both sides we get

$$\int_{-1}^1 \frac{d}{dx}(fg) = \int_{-1}^1 g\frac{df}{dx} + \int_{-1}^1 f\frac{dg}{dx}$$

$$\implies f(1)g(1) - f(-1)g(-1) = \int_{-1}^1 g\frac{df}{dx} + \int_{-1}^1 f\frac{dg}{dx}$$

# Derivative transfer

Note that

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

Integrating both sides we get

$$\int_{-1}^1 \frac{d}{dx}(fg) = \int_{-1}^1 g\frac{df}{dx} + \int_{-1}^1 f\frac{dg}{dx}$$

$$\implies f(1)g(1) - f(-1)g(-1) = \int_{-1}^1 g\frac{df}{dx} + \int_{-1}^1 f\frac{dg}{dx}$$

Thus if

$$f(1)g(1) - f(-1)g(-1) = 0$$

then we get

$$\int_{-1}^1 g\frac{df}{dx} = - \int_{-1}^1 f\frac{dg}{dx}$$

# Derivative transfer

Note that

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

Integrating both sides we get

$$\int_{-1}^1 \frac{d}{dx}(fg) = \int_{-1}^1 g\frac{df}{dx} + \int_{-1}^1 f\frac{dg}{dx}$$

$$\implies f(1)g(1) - f(-1)g(-1) = \int_{-1}^1 g\frac{df}{dx} + \int_{-1}^1 f\frac{dg}{dx}$$

Thus if

$$f(1)g(1) - f(-1)g(-1) = 0$$

then we get

$$\int_{-1}^1 g\frac{df}{dx} = - \int_{-1}^1 f\frac{dg}{dx}$$

This will be referred to as **derivative-transfer**

# Orthogonality of Legendre polynomials

# Orthogonality of Legendre polynomials

Since  $P_m(x)$  is a polynomial of degree  $m$ , it follows that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

is a basis of the vector space of polynomials  $\mathcal{P}(x)$ .

# Orthogonality of Legendre polynomials

Since  $P_m(x)$  is a polynomial of degree  $m$ , it follows that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

is a basis of the vector space of polynomials  $\mathcal{P}(x)$ .

## Theorem

We have

$$\langle P_m, P_n \rangle = \int_{-1}^1 P_m(x) P_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

*i.e.* Legendre polynomials form an **orthogonal basis** for the vector space  $\mathcal{P}(x)$  and

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

# Orthogonality of Legendre polynomials

The Legendre equation may be written as

$$((1 - x^2)y')' + p(p + 1)y = 0$$

# Orthogonality of Legendre polynomials

The Legendre equation may be written as

$$((1 - x^2)y')' + p(p + 1)y = 0$$

In particular,  $P_m(x)$  satisfies

$$((1 - x^2)P'_m(x))' + m(m + 1)P_m(x) = 0 \quad (*)$$



# Orthogonality of Legendre polynomials

The Legendre equation may be written as

$$((1 - x^2)y')' + p(p + 1)y = 0$$

In particular,  $P_m(x)$  satisfies

$$((1 - x^2)P'_m(x))' + m(m + 1)P_m(x) = 0 \quad (*)$$

## Proof of Orthogonality.

Multiply (\*) by  $P_n$  and integrate to get

$$\int_{-1}^1 ((1 - x^2)P'_m(x))' P_n + m(m + 1) \int_{-1}^1 P_m P_n = 0$$

# Orthogonality of Legendre polynomials

The Legendre equation may be written as

$$((1 - x^2)y')' + p(p + 1)y = 0$$

In particular,  $P_m(x)$  satisfies

$$((1 - x^2)P'_m(x))' + m(m + 1)P_m(x) = 0 \quad (*)$$

## Proof of Orthogonality.

Multiply (\*) by  $P_n$  and integrate to get

$$\int_{-1}^1 ((1 - x^2)P'_m)' P_n + m(m + 1) \int_{-1}^1 P_m P_n = 0$$

By derivative transfer ( $f = (1 - x^2)P'_m$  and  $g = P_n$ ), we get

$$- \int_{-1}^1 (1 - x^2)P'_m P'_n + m(m + 1) \int_{-1}^1 P_m P_n = 0$$

continued ...

Interchanging the roles of  $m$  and  $n$ , we get

$$- \int_{-1}^1 (1-x^2)P'_m P'_n + n(n+1) \int_{-1}^1 P_m P_n = 0$$

continued ...

Interchanging the roles of  $m$  and  $n$ , we get

$$- \int_{-1}^1 (1-x^2)P'_m P'_n + n(n+1) \int_{-1}^1 P_m P_n = 0$$

Subtracting the two identities, we obtain

$$[m(m+1) - n(n+1)] \int_{-1}^1 P_m P_n = 0$$

continued ...

Interchanging the roles of  $m$  and  $n$ , we get

$$-\int_{-1}^1 (1-x^2)P'_m P'_n + n(n+1) \int_{-1}^1 P_m P_n = 0$$

Subtracting the two identities, we obtain

$$[m(m+1) - n(n+1)] \int_{-1}^1 P_m P_n = 0$$

If  $m \neq n$  we get

$$\int_{-1}^1 P_m P_n = 0$$

Thus,  $P_m$  and  $P_n$  are orthogonal. □

# Orthogonality of Legendre polynomials

## Rodrigues formula

It only remains to show that  $\|P_n(x)\|^2 = \frac{2}{2n+1}$ .

# Orthogonality of Legendre polynomials

## Rodrigues formula

It only remains to show that  $\|P_n(x)\|^2 = \frac{2}{2n+1}$ .

We need some intermediate steps before we can show this.

# Orthogonality of Legendre polynomials

## Rodrigues formula

It only remains to show that  $\|P_n(x)\|^2 = \frac{2}{2n+1}$ .

We need some intermediate steps before we can show this.

Denote by  $D$  the differential operator  $\frac{d}{dx}$ .



# Orthogonality of Legendre polynomials

## Rodrigues formula

It only remains to show that  $\|P_n(x)\|^2 = \frac{2}{2n+1}$ .

We need some intermediate steps before we can show this.

Denote by  $D$  the differential operator  $\frac{d}{dx}$ .

Let us first note that for  $0 \leq i < n$

$$(D^i(x^2 - 1)^n)(1) = 0$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

It only remains to show that  $\|P_n(x)\|^2 = \frac{2}{2n+1}$ .

We need some intermediate steps before we can show this.

Denote by  $D$  the differential operator  $\frac{d}{dx}$ .

Let us first note that for  $0 \leq i < n$

$$(D^i(x^2 - 1)^n)(1) = 0$$

This is clear once we observe

$$\begin{aligned} D^i(x^2 - 1)^n &= D^i((x - 1)^n(2 + x - 1)^n) \\ &= D^i(2^n(x - 1)^n + (*) (x - 1)^{n+1} + \dots) \end{aligned}$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

It only remains to show that  $\|P_n(x)\|^2 = \frac{2}{2n+1}$ .

We need some intermediate steps before we can show this.

Denote by  $D$  the differential operator  $\frac{d}{dx}$ .

Let us first note that for  $0 \leq i < n$

$$(D^i(x^2 - 1)^n)(1) = 0$$

This is clear once we observe

$$\begin{aligned} D^i(x^2 - 1)^n &= D^i((x - 1)^n(2 + x - 1)^n) \\ &= D^i(2^n(x - 1)^n + (*) (x - 1)^{n+1} + \dots) \end{aligned}$$

By the same reasoning we get for  $0 \leq i < n$

$$(D^i(x^2 - 1)^n)(-1) = 0$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

It only remains to show that  $\|P_n(x)\|^2 = \frac{2}{2n+1}$ .

We need some intermediate steps before we can show this.

Denote by  $D$  the differential operator  $\frac{d}{dx}$ .

Let us first note that for  $0 \leq i < n$

$$(D^i(x^2 - 1)^n)(1) = 0$$

This is clear once we observe

$$\begin{aligned} D^i(x^2 - 1)^n &= D^i((x-1)^n(2+x-1)^n) \\ &= D^i(2^n(x-1)^n + (*) (x-1)^{n+1} + \dots) \end{aligned}$$

By the same reasoning we get for  $0 \leq i < n$

$$(D^i(x^2 - 1)^n)(-1) = 0$$

Consider the polynomial of degree  $n$  given by

$$y(x) = D^n(x^2 - 1)^n$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

For  $k < n$  consider the integral

$$\int_{-1}^1 P_k(x)y(x) = \int_{-1}^1 P_k(x)D(D^{n-1}(x^2 - 1)^n)$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

For  $k < n$  consider the integral

$$\int_{-1}^1 P_k(x)y(x) = \int_{-1}^1 P_k(x)D(D^{n-1}(x^2 - 1)^n)$$

applying derivative transfer with  $f = D^{n-1}(x^2 - 1)^n$  and  $g = P_k(x)$  we get

# Orthogonality of Legendre polynomials

## Rodrigues formula

For  $k < n$  consider the integral

$$\int_{-1}^1 P_k(x)y(x) = \int_{-1}^1 P_k(x)D(D^{n-1}(x^2 - 1)^n)$$

applying derivative transfer with  $f = D^{n-1}(x^2 - 1)^n$  and  $g = P_k(x)$  we get

$$\begin{aligned}\int_{-1}^1 P_k(x)y(x) &= - \int_{-1}^1 DP_k(x)D^{n-1}(x^2 - 1)^n \\ &= \int_{-1}^1 D^2 P_k(x)D^{n-2}(x^2 - 1)^n \\ &= \int_{-1}^1 D^n P_k(x)(x^2 - 1)^n = 0\end{aligned}$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

For  $k < n$  consider the integral

$$\int_{-1}^1 P_k(x)y(x) = \int_{-1}^1 P_k(x)D(D^{n-1}(x^2 - 1)^n)$$

applying derivative transfer with  $f = D^{n-1}(x^2 - 1)^n$  and  $g = P_k(x)$  we get

$$\begin{aligned}\int_{-1}^1 P_k(x)y(x) &= - \int_{-1}^1 DP_k(x)D^{n-1}(x^2 - 1)^n \\ &= \int_{-1}^1 D^2 P_k(x)D^{n-2}(x^2 - 1)^n \\ &= \int_{-1}^1 D^n P_k(x)(x^2 - 1)^n = 0\end{aligned}$$

We have repeatedly applied derivative transfer with  $f = D^{n-i}(x^2 - 1)^n$  and  $g = D^{i-1}P_k(x)$ .



# Orthogonality of Legendre polynomials

## Rodrigues formula

For  $k < n$  consider the integral

$$\int_{-1}^1 P_k(x)y(x) = \int_{-1}^1 P_k(x)D(D^{n-1}(x^2 - 1)^n)$$

applying derivative transfer with  $f = D^{n-1}(x^2 - 1)^n$  and  $g = P_k(x)$  we get

$$\begin{aligned}\int_{-1}^1 P_k(x)y(x) &= - \int_{-1}^1 DP_k(x)D^{n-1}(x^2 - 1)^n \\ &= \int_{-1}^1 D^2 P_k(x)D^{n-2}(x^2 - 1)^n \\ &= \int_{-1}^1 D^n P_k(x)(x^2 - 1)^n = 0\end{aligned}$$

We have repeatedly applied derivative transfer with  $f = D^{n-i}(x^2 - 1)^n$  and  $g = D^{i-1}P_k(x)$ .

Since  $P_k(x)$  is a polynomial of degree  $k$  we get that  $D^n P_k(x) = 0$ .

# Orthogonality of Legendre polynomials

## Rodrigues formula

This forces that  $y(x) = cP_n(x)$  for some nonzero constant  $c$  as we know that  $P_k(x)$ 's are orthogonal to each other.

# Orthogonality of Legendre polynomials

## Rodrigues formula

This forces that  $y(x) = cP_n(x)$  for some nonzero constant  $c$  as we know that  $P_k(x)$ 's are orthogonal to each other.

$$\begin{aligned}D^n(x^2 - 1)^n &= D^n((x - 1)^n(2 + x - 1)^n) \\ &= D^n(2^n(x - 1)^n + (*) (x - 1)^{n+1} + \dots)\end{aligned}$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

This forces that  $y(x) = cP_n(x)$  for some nonzero constant  $c$  as we know that  $P_k(x)$ 's are orthogonal to each other.

$$\begin{aligned}D^n(x^2 - 1)^n &= D^n((x - 1)^n(2 + x - 1)^n) \\ &= D^n(2^n(x - 1)^n + (*) (x - 1)^{n+1} + \dots)\end{aligned}$$

From the above it is clear that

$$y(1) = n!2^n$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

This forces that  $y(x) = cP_n(x)$  for some nonzero constant  $c$  as we know that  $P_k(x)$ 's are orthogonal to each other.

$$\begin{aligned}D^n(x^2 - 1)^n &= D^n((x - 1)^n(2 + x - 1)^n) \\ &= D^n(2^n(x - 1)^n + (*) (x - 1)^{n+1} + \dots)\end{aligned}$$

From the above it is clear that

$$y(1) = n!2^n$$

Thus, we can normalize our Legendre polynomials so that  $P_m(1) = 1$ . That is, take

$$P_m(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

# Orthogonality of Legendre polynomials

## Rodrigues formula

This forces that  $y(x) = cP_n(x)$  for some nonzero constant  $c$  as we know that  $P_k(x)$ 's are orthogonal to each other.

$$\begin{aligned}D^n(x^2 - 1)^n &= D^n((x - 1)^n(2 + x - 1)^n) \\ &= D^n(2^n(x - 1)^n + (*) (x - 1)^{n+1} + \dots)\end{aligned}$$

From the above it is clear that

$$y(1) = n!2^n$$

Thus, we can normalize our Legendre polynomials so that  $P_m(1) = 1$ . That is, take

$$P_m(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

This is called **Rodrigues formula**.

# Orthogonality of Legendre polynomials

Computing  $\|P_n(x)\|$

Proof.

$$\int_{-1}^1 P_n(x)P_n(x) dx = \frac{1}{2^{2n}(n!)^2} \int_{-1}^1 \frac{d^n}{dx^n} (x^2 - 1)^n \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

# Orthogonality of Legendre polynomials

Computing  $\|P_n(x)\|$

Proof.

$$\begin{aligned}\int_{-1}^1 P_n(x)P_n(x) dx &= \frac{1}{2^{2n}(n!)^2} \int_{-1}^1 \frac{d^n}{dx^n}(x^2 - 1)^n \frac{d^n}{dx^n}(x^2 - 1)^n dx \\ &= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^1 (x^2 - 1)^n \frac{d^{2n}}{dx^{2n}}(x^2 - 1)^n dx\end{aligned}$$

by derivative transfer



# Orthogonality of Legendre polynomials

Computing  $\|P_n(x)\|$

Proof.

$$\begin{aligned}\int_{-1}^1 P_n(x)P_n(x) dx &= \frac{1}{2^{2n}(n!)^2} \int_{-1}^1 \frac{d^n}{dx^n}(x^2 - 1)^n \frac{d^n}{dx^n}(x^2 - 1)^n dx \\ &= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^1 (x^2 - 1)^n \frac{d^{2n}}{dx^{2n}}(x^2 - 1)^n dx\end{aligned}$$

by derivative transfer

$$= \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^1 (1 - x^2)^n dx$$

# Orthogonality of Legendre polynomials

Computing  $\|P_n(x)\|$

Proof.

$$\begin{aligned}\int_{-1}^1 P_n(x)P_n(x) dx &= \frac{1}{2^{2n}(n!)^2} \int_{-1}^1 \frac{d^n}{dx^n}(x^2 - 1)^n \frac{d^n}{dx^n}(x^2 - 1)^n dx \\ &= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^1 (x^2 - 1)^n \frac{d^{2n}}{dx^{2n}}(x^2 - 1)^n dx\end{aligned}$$

by derivative transfer

$$= \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^1 (1 - x^2)^n dx$$

$$I_n = \int_{-1}^1 (1 - x^2)^n dx = \int_{-1}^1 (1 - x^2)^n \frac{dx}{dx}$$

$$\stackrel{dt}{=} 2n \int_{-1}^1 (1 - x^2)^{n-1} x^2 dx = -2nI_n + 2nI_{n-1}$$

Proof.

We get the recursive relation

$$(2n + 1)I_n = 2nI_{n-1}$$

Proof.

We get the recursive relation

$$(2n + 1)I_n = 2nI_{n-1}$$

We conclude that

$$I_n = \frac{2n}{2n+1} \frac{2(n-1)}{2n-1} \cdots \frac{2}{3} I_0$$

Proof.

We get the recursive relation

$$(2n + 1)I_n = 2nI_{n-1}$$

We conclude that

$$I_n = \frac{2n}{2n+1} \frac{2(n-1)}{2n-1} \cdots \frac{2}{3} I_0$$

We conclude that

$$\begin{aligned} \|P_n(x)\| &= \frac{(2n)!}{2^{2n}(n!)^2} \frac{2n}{2n+1} \frac{2(n-1)}{2n-1} \cdots \frac{2}{3} I_0 \\ &= \frac{I_0}{2n+1} = \frac{2}{2n+1} \end{aligned}$$



Proof.

We get the recursive relation

$$(2n + 1)I_n = 2nI_{n-1}$$

We conclude that

$$I_n = \frac{2n}{2n+1} \frac{2(n-1)}{2n-1} \cdots \frac{2}{3} I_0$$

We conclude that

$$\begin{aligned} \|P_n(x)\| &= \frac{(2n)!}{2^{2n}(n!)^2} \frac{2n}{2n+1} \frac{2(n-1)}{2n-1} \cdots \frac{2}{3} I_0 \\ &= \frac{I_0}{2n+1} = \frac{2}{2n+1} \end{aligned}$$



This completes the proof of the theorem.

These exercises are related to some facts from linear algebra that we used in the lecture today.

- 1 Recall the proof of the Gram Schmidt orthogonalization lemma.
- 2 Let  $f_i(x)$  (for  $i \geq 0$ ) be a collection of nonzero polynomials. Assume that  $f_i(x)$  has degree  $i$ . Show that  $\{f_0(x), f_1(x), \dots, f_n(x)\}$  is a basis for the vector space consisting of polynomials of degree  $\leq n$ .

The Legendre equation

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

When  $p = m$  is an integer then exactly one of the two independent solutions is a polynomial, denoted by  $P_m(x)$ . This is a polynomial of degree  $m$ .

### Theorem

We have

$$\langle P_m, P_n \rangle = \int_{-1}^1 P_m(x)P_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

*i.e. Legendre polynomials form an **orthogonal basis** for the vector space  $\mathcal{P}(x)$  and*

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$



# Expansion of polynomial in terms of $P_n$ 's

## Expansion of polynomial in terms of $P_n$ 's

Since each  $P_n(x)$  is a polynomial of degree  $n$ , we see that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

form a basis for the vector space of polynomials  $\mathcal{P}(x)$ .

# Expansion of polynomial in terms of $P_n$ 's

Since each  $P_n(x)$  is a polynomial of degree  $n$ , we see that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

form a basis for the vector space of polynomials  $\mathcal{P}(x)$ .

If  $f(x)$  is a polynomial of degree  $n$ , then we can express

$$f(x) = \sum_{k=0}^n a_k P_k(x) \quad a_k \in \mathbb{R}$$

# Expansion of polynomial in terms of $P_n$ 's

Since each  $P_n(x)$  is a polynomial of degree  $n$ , we see that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

form a basis for the vector space of polynomials  $\mathcal{P}(x)$ .

If  $f(x)$  is a polynomial of degree  $n$ , then we can express

$$f(x) = \sum_{k=0}^n a_k P_k(x) \quad a_k \in \mathbb{R}$$

To find  $a_k$ , we can use orthogonality of  $P_n$ 's.

# Expansion of polynomial in terms of $P_n$ 's

Since each  $P_n(x)$  is a polynomial of degree  $n$ , we see that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

form a basis for the vector space of polynomials  $\mathcal{P}(x)$ .

If  $f(x)$  is a polynomial of degree  $n$ , then we can express

$$f(x) = \sum_{k=0}^n a_k P_k(x) \quad a_k \in \mathbb{R}$$

To find  $a_k$ , we can use orthogonality of  $P_n$ 's.

$$\int_{-1}^1 f(x) P_k(x) dx = \int_{-1}^1 \left( \sum_{i=0}^n a_i P_i(x) \right) P_k(x) dx$$

# Expansion of polynomial in terms of $P_n$ 's

Since each  $P_n(x)$  is a polynomial of degree  $n$ , we see that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

form a basis for the vector space of polynomials  $\mathcal{P}(x)$ .

If  $f(x)$  is a polynomial of degree  $n$ , then we can express

$$f(x) = \sum_{k=0}^n a_k P_k(x) \quad a_k \in \mathbb{R}$$

To find  $a_k$ , we can use orthogonality of  $P_n$ 's.

$$\begin{aligned} \int_{-1}^1 f(x) P_k(x) dx &= \int_{-1}^1 \left( \sum_{i=0}^n a_i P_i(x) \right) P_k(x) dx \\ &= \sum_{i=0}^n \left( \int_{-1}^1 a_i P_i(x) P_k(x) dx \right) = a_k \int_{-1}^1 P_k(x) P_k(x) dx \end{aligned}$$

# Expansion of polynomial in terms of $P_n$ 's

Since each  $P_n(x)$  is a polynomial of degree  $n$ , we see that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

form a basis for the vector space of polynomials  $\mathcal{P}(x)$ .

If  $f(x)$  is a polynomial of degree  $n$ , then we can express

$$f(x) = \sum_{k=0}^n a_k P_k(x) \quad a_k \in \mathbb{R}$$

To find  $a_k$ , we can use orthogonality of  $P_n$ 's.

$$\begin{aligned} \int_{-1}^1 f(x) P_k(x) dx &= \int_{-1}^1 \left( \sum_{i=0}^n a_i P_i(x) \right) P_k(x) dx \\ &= \sum_{i=0}^n \left( \int_{-1}^1 a_i P_i(x) P_k(x) dx \right) = a_k \int_{-1}^1 P_k(x) P_k(x) dx \\ \implies a_k &= \frac{2n+1}{2} \int_{-1}^1 f(x) P_k(x) dx \end{aligned}$$

# Square-integrable functions



# Square-integrable functions

A function  $f(x)$  on  $[-1, 1]$  is **square-integrable** if

$$\int_{-1}^1 f(x)f(x)dx < \infty$$

# Square-integrable functions

A function  $f(x)$  on  $[-1, 1]$  is **square-integrable** if

$$\int_{-1}^1 f(x)f(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable.

# Square-integrable functions

A function  $f(x)$  on  $[-1, 1]$  is **square-integrable** if

$$\int_{-1}^1 f(x)f(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable.

The set of all square-integrable functions on  $[-1, 1]$  is a vector space and is denoted by  $L^2([-1, 1])$ .

# Square-integrable functions

A function  $f(x)$  on  $[-1, 1]$  is **square-integrable** if

$$\int_{-1}^1 f(x)f(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable.

The set of all square-integrable functions on  $[-1, 1]$  is a vector space and is denoted by  $L^2([-1, 1])$ .

For square-integrable functions  $f$  and  $g$ , we define their inner product by

$$\langle f, g \rangle := \int_{-1}^1 f(x)g(x)dx$$

# Fourier-Legendre series

The Legendre polynomials no longer form a basis for the vector space  $L^2([-1, 1])$  of square-integrable functions.

The Legendre polynomials no longer form a basis for the vector space  $L^2([-1, 1])$  of square-integrable functions.

In fact, the space  $L^2([-1, 1])$  is **HUGE** in comparison with the space spanned by the Legendre polynomials.

The Legendre polynomials no longer form a basis for the vector space  $L^2([-1, 1])$  of square-integrable functions.

In fact, the space  $L^2([-1, 1])$  is **HUGE** in comparison with the space spanned by the Legendre polynomials.

What do we mean by **HUGE** here?



# Fourier-Legendre series

The Legendre polynomials no longer form a basis for the vector space  $L^2([-1, 1])$  of square-integrable functions.

In fact, the space  $L^2([-1, 1])$  is **HUGE** in comparison with the space spanned by the Legendre polynomials.

What do we mean by **HUGE** here?

The set of rational numbers ( $\mathbb{Q}$ ) is contained in the set of real numbers ( $\mathbb{R}$ ) and is “dense”. However, the size of  $\mathbb{R}$  is much larger than the size of  $\mathbb{Q}$ . These ideas were formulated precisely by Georg Cantor.

But Legendre polynomials form a **maximal orthogonal set** in  $L^2([-1, 1])$ .

# Fourier-Legendre series

But Legendre polynomials form a **maximal orthogonal set** in  $L^2([-1, 1])$ .

This means that a square-integrable function which is orthogonal to all Legendre polynomials is necessarily the constant function “0” (**a nontrivial fact**).

# Fourier-Legendre series

But Legendre polynomials form a **maximal orthogonal set** in  $L^2([-1, 1])$ .

This means that a square-integrable function which is orthogonal to all Legendre polynomials is necessarily the constant function “0” (**a nontrivial fact**).

We can expand any square-integrable function  $f(x)$  on  $[-1, 1]$  in a series of Legendre polynomials

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx$$

This is called the **Fourier-Legendre series** (or simply the **Legendre series**) of  $f(x)$ .

## Theorem (Convergence in norm)

## Theorem (Convergence in norm)

The Fourier-Legendre series of  $f(x) \in L^2([-1, 1])$  given by

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx$$

converges in  $L^2$  norm to  $f(x)$ , that is

$$\|f(x) - \sum_{n=0}^m c_n P_n(x)\| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

## Theorem (Convergence in norm)

The Fourier-Legendre series of  $f(x) \in L^2([-1, 1])$  given by

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx$$

converges in  $L^2$  norm to  $f(x)$ , that is

$$\|f(x) - \sum_{n=0}^m c_n P_n(x)\| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

Pointwise convergence of Fourier-Legendre series to  $f(x)$  is more delicate.

## Theorem (Convergence in norm)

The Fourier-Legendre series of  $f(x) \in L^2([-1, 1])$  given by

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx$$

converges in  $L^2$  norm to  $f(x)$ , that is

$$\|f(x) - \sum_{n=0}^m c_n P_n(x)\| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

Pointwise convergence of Fourier-Legendre series to  $f(x)$  is more delicate.

There are two issues here:

- Does the Fourier-Legendre series converge at  $x$ ?
- If yes, then does it converge to  $f(x)$ ?



A useful result in this direction is the [Legendre expansion theorem](#):

A useful result in this direction is the [Legendre expansion theorem](#):

## Theorem

*If both  $f(x)$  and  $f'(x)$  have at most a finite number of jump discontinuities in the interval  $[-1, 1]$ ,*

A useful result in this direction is the [Legendre expansion theorem](#):

## Theorem

*If both  $f(x)$  and  $f'(x)$  have at most a finite number of jump discontinuities in the interval  $[-1, 1]$ , then the Legendre series [converges to](#)*

A useful result in this direction is the [Legendre expansion theorem](#):

## Theorem

*If both  $f(x)$  and  $f'(x)$  have at most a finite number of jump discontinuities in the interval  $[-1, 1]$ , then the Legendre series **converges to***

$$\frac{1}{2}(f(x_-) + f(x_+)) \quad \text{for } -1 < x < 1$$

A useful result in this direction is the [Legendre expansion theorem](#):

## Theorem

*If both  $f(x)$  and  $f'(x)$  have at most a finite number of jump discontinuities in the interval  $[-1, 1]$ , then the Legendre series **converges to***

$$\frac{1}{2}(f(x_-) + f(x_+)) \quad \text{for } -1 < x < 1$$

$$f(-1_+) \quad \text{for } x = -1$$

# Fourier-Legendre series

A useful result in this direction is the [Legendre expansion theorem](#):

## Theorem

*If both  $f(x)$  and  $f'(x)$  have at most a finite number of jump discontinuities in the interval  $[-1, 1]$ , then the Legendre series **converges to***

$$\frac{1}{2}(f(x_-) + f(x_+)) \quad \text{for } -1 < x < 1$$

$$f(-1_+) \quad \text{for } x = -1$$

$$f(1_-) \quad \text{for } x = 1$$

# Fourier-Legendre series

A useful result in this direction is the [Legendre expansion theorem](#):

## Theorem

*If both  $f(x)$  and  $f'(x)$  have at most a finite number of jump discontinuities in the interval  $[-1, 1]$ , then the Legendre series **converges to***

$$\frac{1}{2}(f(x_-) + f(x_+)) \quad \text{for } -1 < x < 1$$

$$f(-1_+) \quad \text{for } x = -1$$

$$f(1_-) \quad \text{for } x = 1$$

*In particular, the series converges to  $f(x)$  at every point of continuity  $x$ .*

## Example

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$



## Example

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

The Legendre series of  $f(x)$  is

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx$$

## Example

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

The Legendre series of  $f(x)$  is

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx$$

Since  $P_{2n}(x)$  is even function and  $f$  is an odd function, we get

$$c_{2n} = 0 \quad n \geq 0$$

## Example

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

The Legendre series of  $f(x)$  is

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx$$

Since  $P_{2n}(x)$  is even function and  $f$  is an odd function, we get

$$c_{2n} = 0 \quad n \geq 0$$

Recall,  $P_1(x) = x$ , so

$$c_1 = \frac{3}{2} \int_{-1}^1 f(x)x dx = \frac{3}{2}$$

## Example (continued ...)

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ , so

$$c_3 = \frac{7}{2} \int_{-1}^1 f(x) \frac{1}{2}(5x^3 - 3x) dx = \frac{7}{2} \left( \frac{5}{4}x^4 - \frac{3}{2}x^2 \right) \Big|_0^1 = -\frac{7}{8}$$

Check that the Legendre series of  $f$  is

$$\frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \dots$$

## Example (continued ...)

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ , so

$$c_3 = \frac{7}{2} \int_{-1}^1 f(x) \frac{1}{2}(5x^3 - 3x) dx = \frac{7}{2} \left( \frac{5}{4}x^4 - \frac{3}{2}x^2 \right) \Big|_0^1 = -\frac{7}{8}$$

Check that the Legendre series of  $f$  is

$$\frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \dots$$

By the Legendre expansion theorem, this series converges to  $f(x)$  for  $x \neq 0$  and to 0 for  $x = 0$ .

# Least Square Approximation

# Least Square Approximation

## Theorem

*Suppose we want to approximate  $f \in L^2([-1, 1])$  in the sense of least square by polynomials  $p(x)$  of degree  $\leq n$ ;*

# Least Square Approximation

## Theorem

*Suppose we want to approximate  $f \in L^2([-1, 1])$  in the sense of least square by polynomials  $p(x)$  of degree  $\leq n$ ; that is, we want to find a polynomial  $p(x)$  which minimizes*

$$I = \int_{-1}^1 [f(x) - p(x)]^2 dx$$



# Least Square Approximation

## Theorem

*Suppose we want to approximate  $f \in L^2([-1, 1])$  in the sense of least square by polynomials  $p(x)$  of degree  $\leq n$ ; that is, we want to find a polynomial  $p(x)$  which minimizes*

$$I = \int_{-1}^1 [f(x) - p(x)]^2 dx$$

*Then the minimizing polynomial is precisely the first  $n + 1$  terms of the Legendre series of  $f(x)$ , i.e.*

$$c_0 P_0(x) + \dots + c_n P_n(x) \quad c_k = \frac{2k + 1}{2} \int_{-1}^1 f(x) P_k(x) dx$$

# Least Square Approximation

## Theorem

Suppose we want to approximate  $f \in L^2([-1, 1])$  in the sense of least square by polynomials  $p(x)$  of degree  $\leq n$ ; that is, we want to find a polynomial  $p(x)$  which minimizes

$$I = \int_{-1}^1 [f(x) - p(x)]^2 dx$$

Then the minimizing polynomial is precisely the first  $n + 1$  terms of the Legendre series of  $f(x)$ , i.e.

$$c_0P_0(x) + \dots + c_nP_n(x) \quad c_k = \frac{2k+1}{2} \int_{-1}^1 f(x)P_k(x)dx$$

## Proof.

Write degree  $\leq n$  polynomial  $p(x) = \sum_{k=0}^n b_k P_k(x)$ , then

$$I = \int_{-1}^1 \left[ f(x) - \sum_{k=0}^n b_k P_k(x) \right]^2 dx$$

$$\begin{aligned} I &= \int_{-1}^1 \left[ f(x) - \sum_{k=0}^n b_k P_k(x) \right]^2 dx \\ &= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \left[ \int_{-1}^1 f(x) P_k(x) dx \right] \end{aligned}$$

$$\begin{aligned} I &= \int_{-1}^1 \left[ f(x) - \sum_{k=0}^n b_k P_k(x) \right]^2 dx \\ &= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \left[ \int_{-1}^1 f(x) P_k(x) dx \right] \\ &= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \frac{2c_k}{2k+1} \end{aligned}$$

$$\begin{aligned} I &= \int_{-1}^1 \left[ f(x) - \sum_{k=0}^n b_k P_k(x) \right]^2 dx \\ &= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \left[ \int_{-1}^1 f(x) P_k(x) dx \right] \\ &= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \frac{2c_k}{2k+1} \\ &= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} (b_k - c_k)^2 - \sum_{k=0}^n \frac{2}{2k+1} c_k^2 \end{aligned}$$

$$\begin{aligned}
I &= \int_{-1}^1 \left[ f(x) - \sum_{k=0}^n b_k P_k(x) \right]^2 dx \\
&= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \left[ \int_{-1}^1 f(x) P_k(x) dx \right] \\
&= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \frac{2c_k}{2k+1} \\
&= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} (b_k - c_k)^2 - \sum_{k=0}^n \frac{2}{2k+1} c_k^2
\end{aligned}$$

Clearly,  $I$  is minimum when  $b_k = c_k$  for  $k = 0, \dots, n$ .

$$\begin{aligned}
I &= \int_{-1}^1 \left[ f(x) - \sum_{k=0}^n b_k P_k(x) \right]^2 dx \\
&= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \left[ \int_{-1}^1 f(x) P_k(x) dx \right] \\
&= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^n b_k \frac{2c_k}{2k+1} \\
&= \int_{-1}^1 f(x)^2 dx + \sum_{k=0}^n \frac{2}{2k+1} (b_k - c_k)^2 - \sum_{k=0}^n \frac{2}{2k+1} c_k^2
\end{aligned}$$

Clearly,  $I$  is minimum when  $b_k = c_k$  for  $k = 0, \dots, n$ .

**Caution.** If  $f$  has a power series expansion on  $[-1, 1]$ , then best “least square polynomial approximation” to  $f(x)$  is not the partial sums of the power series, in general.



## Some remarks

This brings to an end the discussion of second order linear ODE's which we can solve by power series.

## Some remarks

This brings to an end the discussion of second order linear ODE's which we can solve by power series.

Before we go on to more complicated ODE's, let us review what we have done so far.

## Some remarks

This brings to an end the discussion of second order linear ODE's which we can solve by power series.

Before we go on to more complicated ODE's, let us review what we have done so far.

1. Given an ODE of the type

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \quad (*)$$

first convert it to the standard form

$$y'' + \frac{F_1(x)}{F_0(x)}y' + \frac{F_2(x)}{F_0(x)}y = 0 \quad (**)$$

## Some remarks

This brings to an end the discussion of second order linear ODE's which we can solve by power series.

Before we go on to more complicated ODE's, let us review what we have done so far.

1. Given an ODE of the type

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \quad (*)$$

first convert it to the standard form

$$y'' + \frac{F_1(x)}{F_0(x)}y' + \frac{F_2(x)}{F_0(x)}y = 0 \quad (**)$$

Let

$$p(x) := \frac{F_1(x)}{F_0(x)} \qquad q(x) := \frac{F_2(x)}{F_0(x)}$$

2. Now find the set

$$U := \{x_0 \in \mathbb{R} \mid p(x), q(x) \text{ are analytic at } x_0\}$$

2. Now find the set

$$U := \{x_0 \in \mathbb{R} \mid p(x), q(x) \text{ are analytic at } x_0\}$$

3. By the existence theorem, for every  $x_0 \in U$ , there will exist two independent solutions to the above ODE, call them  $y_1(x)$  and  $y_2(x)$ , such that both of them will be analytic in an interval  $I$  around  $x_0$ .

2. Now find the set

$$U := \{x_0 \in \mathbb{R} \mid p(x), q(x) \text{ are analytic at } x_0\}$$

3. By the existence theorem, for every  $x_0 \in U$ , there will exist two independent solutions to the above ODE, call them  $y_1(x)$  and  $y_2(x)$ , such that both of them will be analytic in an interval  $I$  around  $x_0$ .

4. To find the solutions in a neighborhood of  $x_0$ , set  $y(x) = \sum_{n \geq 0} a_n (x - x_0)^n$  into the ODE (\*) or (\*\*) and get recursive relations involving the  $a_n$ . Note that when you do this, the coefficient functions ( $p(x), q(x), F_0(x), \dots$ ) have to be written as power series in  $x - x_0$ . **Note that the recursive relation you get, will be same, irrespective of whether you choose equation (\*) or (\*\*).**

5. Thus, depending on the situation, you may want to choose (\*) or (\*\*).



5. Thus, depending on the situation, you may want to choose (\*) or (\*\*).

For example, the Legendre equation, in the open interval  $(-1, 1)$  around  $x_0 = 0$ , the equation (\*) looks like

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

while (\*\*) looks like

$$y'' - 2\left(\sum_{n \geq 0} x^{2n+1}\right)y' + p(p + 1)\left(\sum_{n \geq 0} x^{2n}\right)y = 0$$

In this case it is clear that, we should choose (\*), as it will be easier to work with. This is what we did in class.